Loans & investments with regular payments

定期付款的贷款与投资

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** List the numbers in the words: the loan or savings amount, the payment, the yearly rate, how often interest is compounded (C/Y), how often you pay (P/Y), and how many years.
   > 把题目里的数列出来：贷款额或储蓄额、每期付款、年利率、每年计息几次（C/Y）、每年付款几次（P/Y）、多少年。
2. 🎯 **Asked · 求什么** Which one is blank: the payment PMT, the loan size PV, or the savings total FV? Does the question also want the total paid or the interest?
   > 空着的是哪一个：每期付款 PMT、贷款额 PV，还是储蓄终值 FV？题目是否还要还款总额或利息？
3. ✏️ **Picture it · 画出来** Draw a timeline: one arrow for each payment at the end of each period. Money today sits at the left (PV); money at the end sits at the right (FV).
   > 画一条时间线：每期末画一个付款箭头。今天的钱在左端（PV），最后的钱在右端（FV）。
4. 🧰 **Tool · 用什么工具** i = rate per PAYMENT period, n = number of payments. Saving: FV formula. Loan size: PV formula. Loan payment: PMT formula. Or fill the TVM list, money out negative.
   > i＝每个「付款期」的利率，n＝付款次数。储蓄用 FV 公式；求能借多少用 PV 公式；求月供用 PMT 公式。也可以填 TVM 清单，付出的钱为负。
5. ✅ **Check · 检验** Payments times n must be MORE than the loan (you pay interest) and LESS than the savings total (interest is added). The difference is the interest.
   > 每期付款乘以 n：贷款题要「大于」贷款额（你付了利息），储蓄题要「小于」终值（利息加进来了）。两者的差就是利息。

⭐ Must know

必须记住

- Saving with equal deposits → FV. Loan → amount borrowed is PV; solve for PMT.
  > 定期等额存款 → 求终值 FV。贷款 → 借款额是现值 PV，求每期还款 PMT。
- i = rate per PAYMENT period, n = number of payments (monthly, 5 years → $n = 60$).
  > i＝每个还款期的利率，n＝还款次数（按月 5 年 → $n = 60$）。
- Interest = total paid − amount borrowed (or − total deposited).
  > 利息＝总付款 − 借款额（储蓄则 − 总存入）。

$$FV = PMT \cdot \frac{(1+i)^n - 1}{i}$$

$$PV = PMT \cdot \frac{1 - (1+i)^{-n}}{i}$$

🔍 Watch out for

当心

1. Using the yearly rate as the monthly rate
   > 把年利率当成月利率
2. Counting years instead of payments
   > 数了年数而不是付款次数
3. Calling the total paid the interest
   > 把还款总额当成利息
4. Using r over 12 for a Canadian mortgage
   > 加拿大房贷直接用 r 除以 12
5. Using the savings formula to find a loan
   > 求贷款额却用了储蓄公式

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can turn a yearly percent into a monthly decimal rate and type a power with a negative exponent into my calculator.

> 我能把年利率百分数换成每月的小数利率，也会在计算器上输入负指数的乘方。

(6′)

○ I can list PV or FV, PMT, i and n from the words and say which one is missing.

> 我能从题目文字里列出 PV 或 FV、PMT、i 和 n，并说出缺的是哪一个。

(6′)

○ I can follow, then finish, a loan solution: payment, total paid, interest.

> 我能看懂并补全一道贷款题：月供、还款总额、利息。

(10′)

○ I can find a loan payment and its interest on my own with at most two hints.

> 我能在最多两个提示下，独立求出贷款月供和利息。

(8′)

○ I can spot the five classic annuity mistakes in someone else's work.

> 我能在别人的解答里找出年金题的五个经典错误。

(8′)

○ I can do the Level 1–2 practice questions and say whether each answer is reasonable.

> 我能做出 1–2 级练习题，并判断每个答案是否合理。

(7′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. An account pays 4.8% per year, compounded monthly. What is the monthly rate as a decimal?

> 某账户年利率 4.8%，按月复利。每月利率写成小数是多少?

A. 0.048 B. 0.4 C. 0.004 D. 0.0004

If you missed it · 如果做错了 — **Percent, decimals and rates**

> **百分数、小数与比率**

: Percent to decimal: divide by 100 ($4.5\% = 0.045$). A rate per year split into n periods is $\frac{r}{n}$ per period: 6% compounded monthly is 0.005 per month. Increase by r: multiply by $(1 + r)$.

> 百分数化小数：除以 100（$4.5\% = 0.045$）。年利率分成 n 期，每期是 $\frac{r}{n}$：年利率 6% 按月复利，每月 0.005。增长 r：乘以 $(1 + r)$。

1. $\$200$ increased by 3%: $200 \times 1.03 = 206$.
   > $\$200$ 增长 3%：$200 \times 1.03 = 206$。
2. 3.6% compounded quarterly: $\frac{0.036}{4} = 0.009$ per quarter.
   > 年利率 3.6% 按季复利：每季 $\frac{0.036}{4} = 0.009$。

2. $\$5000$ earns 4% per year, compounded once a year, for 2 years. How much is in the account at the end?

> $\$5000$ 按年利率 4%、每年复利一次存 2 年。最后账户里有多少钱?

A. $\$5400$ B. $\$5408$ C. $\$5800$ D. $\$5200$

If you missed it · 如果做错了 — **Percent, decimals and rates**

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   > 年利率 3.6% 按季复利：每季 $\frac{0.036}{4} = 0.009$。

3. Evaluate 1.02^{-2}, to 4 decimal places.

> 计算 1.02^{-2}，保留 4 位小数。

A. 0.9612 B. -1.0404 C. 1.0404 D. 0.9600

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

> 同底数：相乘→指数相加（$x^3x^4 = x^7$）；相除→指数相减（$x^7 \div x^2 = x^5$）；幂的乘方→指数相乘（$(x^3)^2 = x^6$）。积的乘方每个因数都要乘方：$(2x)^3 = 8x^3$。

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
   > $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$。
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).
   > $\frac{6x^5}{2x^2} = 3x^3$（系数相除，指数相减）。

4. Evaluate $\dfrac{1-1.01^{-12}}{0.01}$, to 2 decimal places.

> 计算 $\dfrac{1-1.01^{-12}}{0.01}$，保留 2 位小数。

A. 11.26 B. 12.68 C. 0.11 D. -11.26

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

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Answers · 答案: 1: C · 2: B · 3: A · 4: A

🔍 Common-errors sheet

常见错误单

1. Using the yearly rate as the monthly rate

> **把年利率当成月利率**

A $\$6000$ loan at 6% per year compounded monthly is repaid monthly over 2 years. Find the payment.

> $\$6000$ 贷款，年利率 6%、按月复利，2 年按月还清。求月供。

1 $i = 0.06,\ n = 24$

> $i = 0.06,\ n = 24$

2 $PMT = \dfrac{6000(0.06)}{1 - 1.06^{-24}} \approx \$478.07$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Counting years instead of payments

> **数了年数而不是付款次数**

You save $\$50$ at the end of every month for 6 years at 3% compounded monthly. Find the future value.

> 你每月月底存 $\$50$，存 6 年，年利率 3% 按月复利。求终值。

1 $i = \dfrac{0.03}{12} = 0.0025,\ n = 6$

> $i = \dfrac{0.03}{12} = 0.0025,\ n = 6$

2 $FV = 50 \cdot \dfrac{1.0025^{6} - 1}{0.0025} \approx \$301.88$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Calling the total paid the interest

> **把还款总额当成利息**

A $\$10\,000$ loan at 5% per year compounded monthly is repaid with monthly payments of $\$299.71$ for 3 years. How much interest is paid?

> $\$10\,000$ 贷款，年利率 5%、按月复利，每月还 $\$299.71$，还 3 年。一共付了多少利息?

1 Total of the payments: $36 \times 299.71 = \$10\,789.56$

> 还款总额：$36 \times 299.71 = \$10\,789.56$

2 Interest $= \$10\,789.56$

> 利息 $= \$10\,789.56$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Using r over 12 for a Canadian mortgage

> **加拿大房贷直接用 r 除以 12**

A $\$250\,000$ mortgage at 6% per year compounded semi-annually is amortized over 25 years with monthly payments. Find the payment.

> $\$250\,000$ 房贷，年利率 6%、半年复利，25 年摊还，按月还款。求月供。

1 $i = \dfrac{0.06}{12} = 0.005,\ n = 300$

> $i = \dfrac{0.06}{12} = 0.005,\ \ n = 300$

2 $PMT = \dfrac{250\,000(0.005)}{1 - 1.005^{-300}} \approx \$1610.75$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Using the savings formula to find a loan

> **求贷款额却用了储蓄公式**

You can afford $\$400$ a month for 5 years. At 6% compounded monthly, what is the largest loan you can take?

> 你每月能还 $\$400$，还 5 年。年利率 6% 按月复利，最多能借多少？

1 Use $FV = 400 \cdot \dfrac{1.005^{60} - 1}{0.005}$

> 用 $FV = 400 \cdot \dfrac{1.005^{60} - 1}{0.005}$

2 $\approx \$27\,908.01$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 1 —** The question shows "6%", so 0.06 goes straight in. But interest is added every MONTH, so each month gets only one twelfth of it.

   > 题目写着「6%」，就直接代了 0.06。可是利息每「月」算一次，每个月只有它的十二分之一。

   $i = 0.005$, $n = 24$: $PMT = \dfrac{6000(0.005)}{1 - 1.005^{-24}} \approx \$265.92$. (Sense check: $6000 \div 24 = 250$, so 478 is far too high.)

   > $i = 0.005$，$n = 24$：$PMT = \dfrac{6000(0.005)}{1 - 1.005^{-24}} \approx \$265.92$。（合理性：$6000 \div 24 = 250$，$478$ 明显太高。）

2. **line · 第 1 —** "6 years" is the number in the question, so it becomes n. But n counts deposits, and there is one every month.

   > 题目里的数是「6 年」，于是就成了 n。可是 n 数的是存款次数，每月存一次。

   $n = 12 \times 6 = 72$: $FV = 50 \cdot \dfrac{1.0025^{72} - 1}{0.0025} \approx \$3938.97$. (You deposit $\$3600$ yourself, so $\$301.88$ cannot be right.)

   > $n = 12 \times 6 = 72$：$FV = 50 \cdot \dfrac{1.0025^{72} - 1}{0.0025} \approx \$3938.97$。（你自己就存了 $\$3600$，所以 $\$301.88$ 不可能对。）

3. **line · 第 2 —** All that money leaves your account, so it feels like the cost. But most of it is paying back the $\$10\,000$ you borrowed; only the extra is interest.

   > 这些钱都从你账户里出去了，所以感觉全是成本。可其中大部分是在还你借的 $\$10\,000$，只有多出来的部分才是利息。

   Interest $= 10\,789.56 - 10\,000 = \$789.56$.

   > 利息 $= 10\,789.56 - 10\,000 = \$789.56$。

4. **line · 第 1 —** "Paid monthly, so divide by 12" works only when interest is ALSO added monthly. Here interest is added twice a year, so the monthly rate must grow to 3% over 6 months.

   > 「按月还就除以 12」只在利息「也」按月计算时成立。这里每年只计息两次，所以月利率要满足：6 个月累积起来正好是 3%。

   $i = 1.03^{1/6} - 1 \approx 0.0049386$, so $PMT \approx \$1599.52$. On a TVM solver: P/Y $= 12$, C/Y $= 2$.

   > $i = 1.03^{1/6} - 1 \approx 0.0049386$，所以 $PMT \approx \$1599.52$。用 TVM：P/Y $= 12$，C/Y $= 2$。

5. **line · 第 1 —** Both formulas look alike, and FV is the one used most. But a loan is money you get TODAY, at the left end of the timeline, so it is a present value.

   > 两个公式长得很像，FV 又最常用。可是贷款是你「今天」拿到的钱，在时间线左端，所以是现值。

   $PV = 400 \cdot \dfrac{1 - 1.005^{-60}}{0.005} \approx \$20\,690.22$. (Sense check: you pay $400 \times 60 = \$24\,000$ in total, so the loan must be LESS than that.)

   > $PV = 400 \cdot \dfrac{1 - 1.005^{-60}}{0.005} \approx \$20\,690.22$。（合理性：你一共还 $400 \times 60 = \$24\,000$，所以贷款额一定「小于」它。）