Simple trig equations

简单三角方程

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** The interval (usually $0^\circ \le \theta < 360^\circ$) and the equation. Isolate the ratio: $\sin\theta$, $\cos\theta$ or $\tan\theta$ equals a number. If sine or cosine is outside $[-1, 1]$, stop: no solution.
   > 看区间（通常是 $0^\circ \le \theta < 360^\circ$）和方程。先把三角比分离出来：$\sin\theta$、$\cos\theta$ 或 $\tan\theta$ 等于一个数。若 sin 或 cos 不在 $[-1, 1]$ 内，停：无解。
2. 🎯 **Asked · 求什么** ALL the angles in the interval, exact if the value is special, otherwise rounded as asked. Usually two; 0 and ± 1 give angles on the axes.
   > 区间内的「所有」角：特殊值写精确角，否则按要求取近似。通常有两个；0 和 ± 1 对应坐标轴上的角。
3. ✏️ **Picture it · 画出来** A quick unit circle: mark where the ratio is positive or negative (CAST), then draw two terminal arms with the same reference angle.
   > 画一个简单的单位圆：用 CAST 标出该比值为正或为负的象限，再画两条参考角相同的终边。
4. 🧰 **Tool · 用什么工具** Reference angle from the POSITIVE value (special triangle or inverse key). Then Q I: θ_R, Q II: $180^\circ - \theta_R$, Q III: $180^\circ + \theta_R$, Q IV: $360^\circ - \theta_R$.
   > 用「正值」求参考角（特殊三角形或反三角键）。然后第一象限 θ_R，第二象限 $180^\circ - \theta_R$，第三象限 $180^\circ + \theta_R$，第四象限 $360^\circ - \theta_R$。
5. ✅ **Check · 检验** Put each angle into the ORIGINAL equation (calculator in degrees). Count them: did you find both?
   > 把每个角代回「原」方程（计算器用角度制）。数一数：两个都找到了吗?

⭐ Must know

必须记住

- Isolate the ratio first; $|\sin\theta|$ or $|\cos\theta| > 1$ → no solution.
  > 先把三角比单独放一边；$|\sin\theta|$ 或 $|\cos\theta| > 1$ → 无解。
- Reference angle from the POSITIVE value; the sign picks two quadrants (CAST).
  > 参考角用正值求；符号决定两个象限（CAST）。
- 0 and ± 1 are quadrantal: read them off the unit circle.
  > 0 和 ± 1 对应轴上角：直接从单位圆读。

QI: θ_R QII: $180^\circ - \theta_R$ QIII: $180^\circ + \theta_R$ QIV: $360^\circ - \theta_R$

🔍 Watch out for

当心

1. Stopping at the calculator answer
   > 停在计算器给的答案上
2. Taking the reference angle from the negative value
   > 用负值求参考角
3. Choosing the wrong pair of quadrants
   > 象限选错了
4. Dividing before moving the constant
   > 还没移项就先除
5. Missing that the value is out of range
   > 没发现比值超出范围

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can solve a two-step linear equation, use $\sin^{-1}$ in degree mode, and say where x or y is negative.
  > 我会解两步的一次方程，会在角度制下用 $\sin^{-1}$，知道 x、y 在哪里为负。

  (5′)
- ○ I can ask the five starting questions for a trig equation.
  > 我能对三角方程问出五个入手问题。

  (5′)
- ○ I can follow, then finish, an isolate–reference angle–quadrants–check solution.
  > 我能看懂并补全「分离—参考角—定象限—检验」的解法。

  (8′)
- ○ I can find BOTH angles on my own with at most two hints.
  > 我能在最多两个提示下独立求出「两个」角。

  (7′)
- ○ I can spot the five classic trig-equation mistakes.
  > 我能找出三角方程的五个经典错误。

  (8′)
- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。

  (10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Solve $3y + 2 = -4$.

> 解 $3y + 2 = -4$。

A. $y = -\frac{2}{3}$ B. $y = -2$ C. $y = 2$ D. $y = -6$

If you missed it · 如果做错了 — **Solving a linear equation**

> **解一元一次方程**

: Do the same thing to both sides. Clear brackets, collect x on one side and numbers on the other, then divide. With fractions, multiply every term by the common denominator first.

> 两边做同样的事。先去括号，把含 x 的项移到一边、数字移到另一边，最后除以系数。有分数时先把每一项都乘以公分母。

1. $3(x - 2) = x + 8 \to 3x - 6 = x + 8 \to 2x = 14 \to x = 7$.
   > $3(x - 2) = x + 8 \to 3x - 6 = x + 8 \to 2x = 14 \to x = 7$。
2. $\frac{x}{2} + \frac{x}{3} = 5$: multiply every term by 6 → $3x + 2x = 30 \to x = 6$.
   > $\frac{x}{2} + \frac{x}{3} = 5$：每项乘 6 → $3x + 2x = 30 \to x = 6$。
3. Check by substituting: $3(7 - 2) = 15$ and $7 + 8 = 15$ ✓.
   > 代回检验：$3(7 - 2) = 15$，$7 + 8 = 15$ ✓。

2. Solve $\sqrt{2}\,s + 1 = 0$ for s.

> 由 $\sqrt{2}\,s + 1 = 0$ 解出 s。

A. $-\sqrt{2}$ B. $\frac{1}{\sqrt{2}}$ C. $-\frac{1}{\sqrt{2}}$ D. $-1 - \sqrt{2}$

If you missed it · 如果做错了 — **Rearranging a formula**

> **公式变形（求某个字母）**

: Treat the other letters as numbers and undo operations in reverse order. $A = P(1 + r)$ for r: divide by P, then subtract 1: $r = \frac{A}{P} - 1$.

> 把其他字母当成数字，按相反顺序撤销运算。由 $A = P(1 + r)$ 求 r：先除以 P，再减 1：$r = \frac{A}{P} - 1$。

1. $y = a(x - p)^2 + q$ for a: subtract q, divide by $(x - p)^2$: $a = \frac{y - q}{(x - p)^2}$.
   > 由 $y = a(x - p)^2 + q$ 求 a：减 q，再除以 $(x - p)^2$：$a = \frac{y - q}{(x - p)^2}$。
2. Substitute numbers first if the question gives them — fewer letters, fewer slips.
   > 题目给了数字就先代入——字母越少，越不容易错。

3. With the calculator in degrees, $\sin^{-1}(0.5) =$

> 计算器在角度制下，$\sin^{-1}(0.5) =$

A. 30° B. 60° C. 0.52° D. 45°

If you missed it · 如果做错了 — **SOH CAH TOA and Pythagoras**

> **直角三角形：SOH CAH TOA 与勾股定理**

: In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.

> 在直角三角形中，从角 θ 看：$\sin\theta = \frac{\text{对边}}{\text{斜边}}$，$\cos\theta = \frac{\text{邻边}}{\text{斜边}}$，$\tan\theta = \frac{\text{对边}}{\text{邻边}}$。斜边对着直角；$a^2 + b^2 = c^2$。计算器设为角度（DEG）模式。

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
   > 两直角边 3 和 4：斜边 $\sqrt{9+16} = 5$；对着 3 的角 $\sin\theta = \frac{3}{5}$。
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9^\circ$.
   > 求角：$\theta = \sin^{-1}(0.6) \approx 36.9^\circ$。

4. In which two quadrants is the x-coordinate negative?

> 在哪两个象限里，x 坐标为负?

A. I and II · 第一、二象限 B. I and IV · 第一、四象限 C. III and IV · 第三、四象限 D. II and III · 第二、三象限

If you missed it · 如果做错了 — **Reading a graph: points, intercepts, slope**

> **读图：点、截距、斜率**

: A point is (x, y): across first, then up. x-intercepts are where $y = 0$ (the graph crosses the horizontal axis); the y-intercept is where $x = 0$. Slope $= \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

> 点写作 (x, y)：先横后纵。x 截距是 $y = 0$ 的地方（图象穿过横轴处）；y 截距是 $x = 0$ 的地方。斜率 $= \frac{\text{纵向变化}}{\text{横向变化}} = \frac{y_2 - y_1}{x_2 - x_1}$。

1. For $y = x^2 - 4$: y-intercept at $x = 0$ is -4; x-intercepts solve $x^2 - 4 = 0$: $x = \pm 2$.
   > $y = x^2 - 4$：令 $x = 0$ 得 y 截距 -4；令 $y = 0$ 解 $x^2 - 4 = 0$ 得 x 截距 $x = \pm 2$。
2. Above the x-axis means $y > 0$; below means $y < 0$.
   > 在 x 轴上方表示 $y > 0$；下方表示 $y < 0$。

Answers · 答案: 1: B · 2: C · 3: A · 4: D

🔍 Common-errors sheet

常见错误单

1. Stopping at the calculator answer

> **停在计算器给的答案上**

Solve $\sin\theta = 0.6$ for $0^\circ \le \theta < 360^\circ$, to the nearest tenth.

> 解 $\sin\theta = 0.6$，$0^\circ \le \theta < 360^\circ$，精确到 0.1°。

1 $\theta = \sin^{-1} 0.6 \approx 36.9^\circ$

2 Answer: $\theta \approx 36.9^\circ$.

> 答案：$\theta \approx 36.9^\circ$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

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2. Taking the reference angle from the negative value

> **用负值求参考角**

Solve $\cos\theta = -0.4$ for $0^\circ \le \theta < 360^\circ$, to the nearest tenth.

> 解 $\cos\theta = -0.4$，$0^\circ \le \theta < 360^\circ$，精确到 0.1°。

1 $\theta_R = \cos^{-1}(-0.4) \approx 113.6^\circ$

2 Q II: $180^\circ - 113.6^\circ = 66.4^\circ$; Q III: $180^\circ + 113.6^\circ = 293.6^\circ$.

> 第二象限：$180^\circ - 113.6^\circ = 66.4^\circ$；第三象限：$180^\circ + 113.6^\circ = 293.6^\circ$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

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3. Choosing the wrong pair of quadrants

> **象限选错了**

Solve $\tan\theta = -1$ for $0^\circ \le \theta < 360^\circ$.

> 解 $\tan\theta = -1$，$0^\circ \le \theta < 360^\circ$。

1 $\theta_R = 45^\circ$

2 Tangent is negative in Q II and Q III

> tan 在第二、三象限为负

3 $\theta = 135^\circ,\ 225^\circ$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

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4. Dividing before moving the constant

> **还没移项就先除**

Solve $2\cos\theta - 1 = 0$ for $0° \le \theta < 360°$.

> 解 $2\cos\theta - 1 = 0$，$0° \le \theta < 360°$。

1 Divide by 2: $\cos\theta - 1 = 0$

> 除以 2：$\cos\theta - 1 = 0$

2 $\cos\theta = 1$, so $\theta = 0°$.

> $\cos\theta = 1$，所以 $\theta = 0°$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Missing that the value is out of range

> **没发现比值超出范围**

Solve $4\sin\theta + 7 = 2$ for $0° \le \theta < 360°$.

> 解 $4\sin\theta + 7 = 2$，$0° \le \theta < 360°$。

1 $4\sin\theta = -5$

2 $\sin\theta = -\frac{4}{5}$

3 $\theta_R \approx 53.1°$, so $\theta \approx 233.1°,\ 306.9°$.

> $\theta_R \approx 53.1°$，所以 $\theta \approx 233.1°,\ 306.9°$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** The inverse key returns ONE angle. Sine is positive in Q II as well, so a second angle has the same sine.

   反三角键只给「一个」角。sin 在第二象限也为正，所以还有一个角的 sin 相同。

   Q II: $180^\circ - 36.87^\circ = 143.13^\circ$. Answers: 36.9° and 143.1°.

   第二象限：$180^\circ - 36.87^\circ = 143.13^\circ$。答案：$36.9^\circ$ 和 143.1°。

2. **line · 第 1 —** A reference angle is always between 0° and 90°. Using the negative value gives an angle already in Q II, and the quadrant rules then move it twice.

   参考角永远在 0° 到 90° 之间。用负值求出的角本身已在第二象限，再套象限公式就挪了两次。

   $\theta_R = \cos^{-1} 0.4 \approx 66.42^\circ$. Q II: 113.6°; Q III: 246.4°.

   $\theta_R = \cos^{-1} 0.4 \approx 66.42^\circ$。第二象限：$113.6^\circ$；第三象限：$246.4^\circ$。

3. **line · 第 2 —** In Q III both x and y are negative, so $\tan = \frac{y}{x}$ is positive (the T in CAST). Tangent is negative in Q II and Q IV.

   第三象限 x、y 都为负，所以 $\tan = \frac{y}{x}$ 为正（CAST 里的 T）。tan 在第二、四象限为负。

   Q II: $180^\circ - 45^\circ = 135^\circ$; Q IV: $360^\circ - 45^\circ = 315^\circ$.

   第二象限：$180^\circ - 45^\circ = 135^\circ$；第四象限：$360^\circ - 45^\circ = 315^\circ$。

4. **line · 第 1 —** Dividing by 2 must divide EVERY term: $\frac{2\cos\theta - 1}{2} = \cos\theta - \frac{1}{2}$. Moving the constant first avoids the trap.

   除以 2 要除「每一项」：$\frac{2\cos\theta - 1}{2} = \cos\theta - \frac{1}{2}$。先移常数项就不会掉进这个坑。

   $2\cos\theta = 1$, $\cos\theta = \frac{1}{2}$, $\theta_R = 60^\circ$; cosine positive in Q I and Q IV: 60°, 300°.

   $2\cos\theta = 1$，$\cos\theta = \frac{1}{2}$，$\theta_R = 60^\circ$；cos 在第一、四象限为正：60°，300°。

5. **line · 第 2 —** Dividing by 4 gives $-\frac{5}{4}$, not $-\frac{4}{5}$. The flipped fraction looked "normal", so the impossible value was never noticed.

   除以 4 得 $-\frac{5}{4}$，不是 $-\frac{4}{5}$。分数倒过来后看着很「正常」，于是没发现这是个不可能的值。

   $\sin\theta = -1.25 < -1$. Sine is never below -1: no solution.

   $\sin\theta = -1.25 < -1$。sin 不会小于 -1：无解。