Special angles & the unit circle

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** Which angle, which ratio? Note where the angle sits: between which two axes ($0°$, $90°$, $180°$, $270°$, $360°$)?
2. 🎯 **Asked** An EXACT value: fractions and roots, no decimals. For a point on the unit circle, both coordinates.
3. ✏️ **Picture it** Sketch the terminal arm, drop a line to the x-axis, and draw the special triangle it matches: 45-45-90 with $1, 1, \sqrt{2}$, or 30-60-90 with $1, \sqrt{3}, 2$.
4. 🧰 **Tool** Reference angle (the gap to the x-axis) → value from the triangle with SOH CAH TOA → sign from CAST. On an axis, read the unit-circle point instead.
5. ✅ **Check** Size: sine and cosine are between -1 and 1; $\frac{\sqrt{3}}{2} \approx 0.87$ is the big one, $\frac{1}{2}$ the small one. Does the sign match CAST? A calculator decimal can confirm.

⭐ Must know

- 45-45-90: $1, 1, \sqrt{2}$. 30-60-90: $1, \sqrt{3}, 2$ (1 opposite $30°$).
- Value = reference-angle value with the CAST sign (All, Sin, Tan, Cos in QI to QIV).
- Unit circle point $= (\cos\theta, \sin\theta)$; $\tan 90°$, $\tan 270°$ undefined.

$$\sin 30° = \tfrac{1}{2},\ \cos 30° = \tfrac{\sqrt{3}}{2},\ \tan 30° = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}$$

$$\sin 45° = \cos 45° = \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}}{2},\ \tan 45° = 1$$

🔍 Watch out for

1. Right value, wrong sign
2. Swapping the 30 and 60 sides
3. Measuring the reference angle to the y-axis
4. Mixing up x and y on the unit circle
5. Calling a division by zero zero

🌱 Small wins — tick as you go

- ○ I can use Pythagoras and SOH CAH TOA, rewrite $\frac{1}{\sqrt{2}}$, and name the quadrant of a point. (5′)
- ○ I can draw both special triangles from memory and ask the five starting questions. (5′)
- ○ I can follow, then finish, a quadrant–reference angle–triangle–sign solution. (8′)
- ○ I can find an exact value in any quadrant on my own with at most two hints. (6′)
- ○ I can spot the five classic special-angle mistakes. (8′)
- ○ I can do the Level 1–2 practice questions. (10′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. A right triangle has legs 1 and 1. How long is the hypotenuse?
A. 2 B. $\sqrt{2}$ C. 1 D. $\sqrt{3}$

If you missed it — **Perfect squares and square roots**: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.

2. A right triangle has hypotenuse 2 and one leg 1. How long is the other leg?
A. 3 B. $\sqrt{5}$ C. 1 D. $\sqrt{3}$

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1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
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3. In a right triangle the sides are 3 (opposite θ), 4 (adjacent) and 5 (hypotenuse). What is $\cos\theta$?
A. $\frac{3}{5}$ B. $\frac{3}{4}$ C. $\frac{4}{5}$ D. $\frac{5}{4}$

If you missed it — **SOH CAH TOA and Pythagoras**: In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.

4. $\dfrac{1}{\sqrt{2}}$ is the same as:
A. $\frac{\sqrt{2}}{2}$ B. $\sqrt{2}$ C. $2\sqrt{2}$ D. $\frac{1}{2}$

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1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
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5. The point $(-3, -5)$ is in which quadrant?
A. I B. II C. III D. IV

If you missed it — **Reading a graph: points, intercepts, slope**: A point is (x, y): across first, then up. x-intercepts are where $y = 0$ (the graph crosses the horizontal axis); the y-intercept is where $x = 0$. Slope $= \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

1. For $y = x^2 - 4$: y-intercept at $x = 0$ is -4; x-intercepts solve $x^2 - 4 = 0$: $x = \pm 2$.
2. Above the x-axis means $y > 0$; below means $y < 0$.

Answers: 1: B · 2: D · 3: C · 4: A · 5: C

🔍 Common-errors sheet

1. Right value, wrong sign

Find the exact value of $\cos 135^\circ$.

1 Q II, reference angle 45°

2 $\cos 45^\circ = \frac{\sqrt{2}}{2}$

3 So $\cos 135^\circ = \frac{\sqrt{2}}{2}$.

✍ Which line goes wrong first, and why?

2. Swapping the 30 and 60 sides

Find the exact value of $\sin 240^\circ$.

1 Q III, reference angle 60°

2 $\sin 60^\circ = \frac{1}{2}$

3 Sine is negative in Q III, so $\sin 240^\circ = -\frac{1}{2}$.

✍ Which line goes wrong first, and why?

3. Measuring the reference angle to the y-axis

Find the exact value of $\tan 330^\circ$.

1 330° is in Q IV

2 Reference angle: $330^\circ - 270^\circ = 60^\circ$

3 $\tan 60^\circ = \sqrt{3}$, negative in Q IV: $-\sqrt{3}$.

✍ Which line goes wrong first, and why?

4. Mixing up x and y on the unit circle

The point at 210° on the unit circle is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Find $\sin 210^\circ$.

1 Sine is the first coordinate

2 So $\sin 210^\circ = -\frac{\sqrt{3}}{2}$.

✍ Which line goes wrong first, and why?

5. Calling a division by zero zero

Find $\tan 270^\circ$.

1 The point at 270° is $(0, -1)$

2 $\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = 0$

✍ Which line goes wrong first, and why?

Key (fold this part back)

1. **line 3 —** The triangle gives lengths, which are always positive. The sign has to come from the quadrant: in Q II, x is negative, so cosine is negative. $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.
2. **line 2 —** The short side 1 is opposite the SMALL angle, 30°. The 60° angle faces $\sqrt{3}$, so $\sin 60^\circ = \frac{\sqrt{3}}{2}$. $\sin 60^\circ = \frac{\sqrt{3}}{2}$, so $\sin 240^\circ = -\frac{\sqrt{3}}{2}$.
3. **line 2 —** The reference angle is always the gap to the x-axis, never the y-axis. In Q IV that is $360^\circ - \theta$. Reference angle $360^\circ - 330^\circ = 30^\circ$; $\tan 30^\circ = \frac{\sqrt{3}}{3}$, negative in Q IV: $\tan 330^\circ = -\frac{\sqrt{3}}{3}$.
4. **line 1 —** The point is $(\cos\theta, \sin\theta)$: cosine goes with x (across), sine with y (up). Alphabet helps: c before s, x before y. $\sin 210^\circ$ is the y-coordinate: $-\frac{1}{2}$.
5. **line 2 —** $\frac{0}{-1} = 0$, but $\frac{-1}{0}$ has no value: no number times 0 gives -1. $x = 0$, so $\tan 270^\circ$ is undefined (so is $\tan 90^\circ$).