Special angles & the unit circle

特殊角与单位圆

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Which angle, which ratio? Note where the angle sits: between which two axes ($0°$, $90°$, $180°$, $270°$, $360°$)?
   > 哪个角、哪个比？看角落在哪两条坐标轴之间（$0°$、$90°$、$180°$、$270°$、$360°$）。
2. 🎯 **Asked · 求什么** An EXACT value: fractions and roots, no decimals. For a point on the unit circle, both coordinates.
   > 要「精确值」：分数和根号，不要小数。若问单位圆上的点，要两个坐标。
3. ✏️ **Picture it · 画出来** Sketch the terminal arm, drop a line to the x-axis, and draw the special triangle it matches: 45-45-90 with $1, 1, \sqrt{2}$, or 30-60-90 with $1, \sqrt{3}, 2$.
   > 画出终边，向 x 轴作垂线，再画出对应的特殊三角形：45-45-90 边长 $1, 1, \sqrt{2}$，或 30-60-90 边长 $1, \sqrt{3}, 2$。
4. 🧰 **Tool · 用什么工具** Reference angle (the gap to the x-axis) → value from the triangle with SOH CAH TOA → sign from CAST. On an axis, read the unit-circle point instead.
   > 参考角（到 x 轴的夹角）→ 用 SOH CAH TOA 从三角形读数 → 用 CAST 定符号。若终边在坐标轴上，直接读单位圆上的点。
5. ✅ **Check · 检验** Size: sine and cosine are between -1 and 1; $\frac{\sqrt{3}}{2} \approx 0.87$ is the big one, $\frac{1}{2}$ the small one. Does the sign match CAST? A calculator decimal can confirm.
   > 看大小：sin、cos 都在 -1 与 1 之间；$\frac{\sqrt{3}}{2} \approx 0.87$ 是大的，$\frac{1}{2}$ 是小的。符号和 CAST 对得上吗？可以用计算器的小数核对。

⭐ Must know

必须记住

- 45-45-90: $1, 1, \sqrt{2}$. 30-60-90: $1, \sqrt{3}, 2$ (1 opposite $30°$).
  > 45-45-90：$1, 1, \sqrt{2}$。30-60-90：$1, \sqrt{3}, 2$（1 对 $30°$）。
- Value = reference-angle value with the CAST sign (All, Sin, Tan, Cos in QI to QIV).
  > 值 = 参考角的值，再配上 CAST 符号（第一到第四象限：全正、正弦、正切、余弦）。
- Unit circle point $= (\cos\theta, \sin\theta)$; $\tan 90°$, $\tan 270°$ undefined.
  > 单位圆上的点 $= (\cos\theta, \sin\theta)$；$\tan 90°$、$\tan 270°$ 无意义。

$$\sin 30° = \tfrac{1}{2},\ \cos 30° = \tfrac{\sqrt{3}}{2},\ \tan 30° = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}$$

$$\sin 45° = \cos 45° = \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}}{2},\ \tan 45° = 1$$

🔍 Watch out for

当心

1. Right value, wrong sign
   > 数值对了，符号错了
2. Swapping the 30 and 60 sides
   > 把 30° 和 60° 的边弄反
3. Measuring the reference angle to the y-axis
   > 把参考角量到了 y 轴
4. Mixing up x and y on the unit circle
   > 把单位圆上的 x、y 弄混
5. Calling a division by zero zero
   > 把除以零当成零

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can use Pythagoras and SOH CAH TOA, rewrite $\frac{1}{\sqrt{2}}$, and name the quadrant of a point.

  > 我会用勾股定理和 SOH CAH TOA，会改写 $\frac{1}{\sqrt{2}}$，能说出一个点在第几象限。

  (5′)
- ○ I can draw both special triangles from memory and ask the five starting questions.

  > 我能凭记忆画出两个特殊三角形，并问出五个入手问题。

  (5′)
- ○ I can follow, then finish, a quadrant–reference angle–triangle–sign solution.

  > 我能看懂并补全「定象限—参考角—特殊三角形—定符号」的解法。

  (8′)
- ○ I can find an exact value in any quadrant on my own with at most two hints.

  > 我能在最多两个提示下独立求出任意象限角的精确值。

  (6′)
- ○ I can spot the five classic special-angle mistakes.

  > 我能找出特殊角的五个经典错误。

  (8′)
- ○ I can do the Level 1–2 practice questions.

  > 我能做出 1–2 级练习题。

  (10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. A right triangle has legs 1 and 1. How long is the hypotenuse?

直角三角形两条直角边都是 1，斜边多长?

A. 2 B. $\sqrt{2}$ C. 1 D. $\sqrt{3}$

If you missed it · 如果做错了 — **Perfect squares and square roots**

完全平方数与平方根

: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

2. A right triangle has hypotenuse 2 and one leg 1. How long is the other leg?

直角三角形斜边为 2，一条直角边为 1，另一条直角边多长?

A. 3 B. $\sqrt{5}$ C. 1 D. $\sqrt{3}$

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3. In a right triangle the sides are 3 (opposite θ), 4 (adjacent) and 5 (hypotenuse). What is $\cos\theta$?

直角三角形中，θ 的对边 3、邻边 4、斜边 5。$\cos\theta$ 是多少?

A. $\frac{3}{5}$ B. $\frac{3}{4}$ C. $\frac{4}{5}$ D. $\frac{5}{4}$

If you missed it · 如果做错了 — **SOH CAH TOA and Pythagoras**

直角三角形：SOH CAH TOA 与勾股定理

: In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.

在直角三角形中，从角 θ 看：$\sin\theta = \frac{\text{对边}}{\text{斜边}}$，$\cos\theta = \frac{\text{邻边}}{\text{斜边}}$，$\tan\theta = \frac{\text{对边}}{\text{邻边}}$。斜边对着直角；$a^2 + b^2 = c^2$。计算器设为角度（DEG）模式。

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
   两直角边 3 和 4：斜边 $\sqrt{9+16} = 5$；对着 3 的角 $\sin\theta = \frac{3}{5}$。
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.
   求角：$\theta = \sin^{-1}(0.6) \approx 36.9°$。

4. $\frac{1}{\sqrt{2}}$ is the same as:

> $\frac{1}{\sqrt{2}}$ 等于:

A. $\frac{\sqrt{2}}{2}$ B. $\sqrt{2}$ C. $2\sqrt{2}$ D. $\frac{1}{2}$

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

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   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

5. The point $(-3, -5)$ is in which quadrant?

> 点 $(-3, -5)$ 在第几象限?

A. I B. II C. III D. IV

If you missed it · 如果做错了 — **Reading a graph: points, intercepts, slope**

> **读图：点、截距、斜率**

: A point is (x, y): across first, then up. x-intercepts are where $y = 0$ (the graph crosses the horizontal axis); the y-intercept is where $x = 0$. Slope $= \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

> 点写作 (x, y)：先横后纵。x 截距是 $y = 0$ 的地方（图象穿过横轴处）；y 截距是 $x = 0$ 的地方。斜率 $= \frac{\text{纵向变化}}{\text{横向变化}} = \frac{y_2 - y_1}{x_2 - x_1}$。

1. For $y = x^2 - 4$: y-intercept at $x = 0$ is -4; x-intercepts solve $x^2 - 4 = 0$: $x = \pm 2$.
   > $y = x^2 - 4$：令 $x = 0$ 得 y 截距 -4；令 $y = 0$ 解 $x^2 - 4 = 0$ 得 x 截距 $x = \pm 2$。
2. Above the x-axis means $y > 0$; below means $y < 0$.
   > 在 x 轴上方表示 $y > 0$；下方表示 $y < 0$。

Answers · 答案: 1: B · 2: D · 3: C · 4: A · 5: C

🔍 Common-errors sheet

常见错误单

1. Right value, wrong sign

数值对了，符号错了

Find the exact value of $\cos 135^\circ$.

求 $\cos 135^\circ$ 的精确值。

1 Q II, reference angle 45°

第二象限，参考角 45°

2 $\cos 45^\circ = \frac{\sqrt{2}}{2}$

3 So $\cos 135^\circ = \frac{\sqrt{2}}{2}$.

所以 $\cos 135^\circ = \frac{\sqrt{2}}{2}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Swapping the 30 and 60 sides

把 30° 和 60° 的边弄反

Find the exact value of $\sin 240^\circ$.

求 $\sin 240^\circ$ 的精确值。

1 Q III, reference angle 60°

第三象限，参考角 60°

2 $\sin 60^\circ = \frac{1}{2}$

3 Sine is negative in Q III, so $\sin 240^\circ = -\frac{1}{2}$.

第三象限 sin 为负，所以 $\sin 240^\circ = -\frac{1}{2}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Measuring the reference angle to the y-axis

把参考角量到了 y 轴

Find the exact value of $\tan 330^\circ$.

求 $\tan 330^\circ$ 的精确值。

1 330° is in Q IV

330° 在第四象限

2 Reference angle: $330^\circ - 270^\circ = 60^\circ$

参考角：$330^\circ - 270^\circ = 60^\circ$

3 $\tan 60^\circ = \sqrt{3}$, negative in Q IV: $-\sqrt{3}$.

$\tan 60^\circ = \sqrt{3}$，第四象限为负：$-\sqrt{3}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Mixing up x and y on the unit circle

把单位圆上的 x、y 弄混

The point at 210° on the unit circle is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Find $\sin 210^\circ$.

单位圆上 210° 对应的点是 $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$。求 $\sin 210^\circ$。

1 Sine is the first coordinate

sin 是第一个坐标

2 So $\sin 210^\circ = -\frac{\sqrt{3}}{2}$.

所以 $\sin 210^\circ = -\frac{\sqrt{3}}{2}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Calling a division by zero zero

把除以零当成零

Find $\tan 270^\circ$.

求 $\tan 270^\circ$。

1 The point at 270° is $(0, -1)$

270° 对应的点是 $(0, -1)$

2 $\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = 0$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 3 —** The triangle gives lengths, which are always positive. The sign has to come from the quadrant: in Q II, x is negative, so cosine is negative.

   > 三角形给的是长度，永远是正的。符号要看象限：第二象限 x 为负，所以 cos 为负。

   $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.

   > $\cos 135^\circ = -\frac{\sqrt{2}}{2}$。

2. **line · 第 2 —** The short side 1 is opposite the SMALL angle, 30°. The 60° angle faces $\sqrt{3}$, so $\sin 60^\circ = \frac{\sqrt{3}}{2}$.

   > 短边 1 对着「小」角 30°。60° 对着 $\sqrt{3}$，所以 $\sin 60^\circ = \frac{\sqrt{3}}{2}$。

   $\sin 60^\circ = \frac{\sqrt{3}}{2}$, so $\sin 240^\circ = -\frac{\sqrt{3}}{2}$.

   > $\sin 60^\circ = \frac{\sqrt{3}}{2}$，所以 $\sin 240^\circ = -\frac{\sqrt{3}}{2}$。

3. **line · 第 2 —** The reference angle is always the gap to the x-axis, never the y-axis. In Q IV that is $360^\circ - \theta$.

   > 参考角永远是到 x 轴的夹角，不是到 y 轴的。第四象限是 $360^\circ - \theta$。

   Reference angle $360^\circ - 330^\circ = 30^\circ$; $\tan 30^\circ = \frac{\sqrt{3}}{3}$, negative in Q IV: $\tan 330^\circ = -\frac{\sqrt{3}}{3}$.

   > 参考角 $360^\circ - 330^\circ = 30^\circ$；$\tan 30^\circ = \frac{\sqrt{3}}{3}$，第四象限为负：$\tan 330^\circ = -\frac{\sqrt{3}}{3}$。

4. **line · 第 1 —** The point is $(\cos\theta, \sin\theta)$: cosine goes with x (across), sine with y (up). Alphabet helps: c before s, x before y.

   > 点是 $(\cos\theta, \sin\theta)$：cos 对应 x（横），sin 对应 y（竖）。按字母顺序记：c 在 s 前，x 在 y 前。

   $\sin 210^\circ$ is the y-coordinate: $-\frac{1}{2}$.

   > $\sin 210^\circ$ 是 y 坐标：$-\frac{1}{2}$。

5. **line · 第 2 —** $\frac{0}{-1} = 0$, but $\frac{-1}{0}$ has no value: no number times 0 gives -1.

   > $\frac{0}{-1} = 0$，但 $\frac{-1}{0}$ 没有值：没有哪个数乘 0 能得 -1。

   $x = 0$, so $\tan 270^\circ$ is undefined (so is $\tan 90^\circ$).

   > $x = 0$，所以 $\tan 270^\circ$ 无定义（$\tan 90^\circ$ 也一样）。