Trig ratios of any angle

任意角的三角比

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** What do you have: a point, or one ratio plus a clue (a quadrant, or the sign of another ratio)? Write the two of x, y, r you know, with signs.
   > 已知什么：一个点，还是一个比值加一条线索（象限，或另一个比值的符号）？写出已知的 x、y、r 中的两个，带上符号。
2. 🎯 **Asked · 求什么** Which ratios, exact or decimal? And θ in $0^\circ \le \theta < 360^\circ$, rounded as asked.
   > 要哪些比值，精确值还是小数？以及 $0^\circ \le \theta < 360^\circ$ 内的 θ，按要求取近似。
3. ✏️ **Picture it · 画出来** Sketch the axes, put the point in its quadrant, drop a line to the x-axis to make the reference triangle, and label x, y, r.
   > 画坐标轴，把点放进它的象限，向 x 轴作垂线得到参考三角形，标出 x、y、r。
4. 🧰 **Tool · 用什么工具** Missing piece from $x^2 + y^2 = r^2$ (sign from the quadrant, r always positive). Then $\sin = \frac{y}{r}$, $\cos = \frac{x}{r}$, $\tan = \frac{y}{x}$; reference angle from the positive ratio; place it: $180^\circ-$, $180^\circ+$ or $360^\circ-$.
   > 用 $x^2 + y^2 = r^2$ 求缺的那个（符号看象限，r 永远为正）。再写 $\sin = \frac{y}{r}$、$\cos = \frac{x}{r}$、$\tan = \frac{y}{x}$；用正的比值求参考角，再放到象限里：$180^\circ-$、$180^\circ+$ 或 $360^\circ-$。
5. ✅ **Check · 检验** CAST: do the signs of your three ratios match the quadrant? Is θ really in that quadrant?
   > CAST：三个比值的符号和象限对得上吗？θ 真的落在那个象限吗？

⭐ Must know

必须记住

- $r = \sqrt{x^2 + y^2} > 0$; $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$.
  > $r = \sqrt{x^2 + y^2} > 0$；$\sin\theta = \frac{y}{r}$，$\cos\theta = \frac{x}{r}$，$\tan\theta = \frac{y}{x}$。
- One ratio + quadrant: draw the triangle, find the third side, give x, y their quadrant signs.
  > 已知一个比值和象限：画三角形，求第三边，按象限给 x, y 定符号。
- Angle: reference angle from the positive ratio, then place it in the quadrant.
  > 求角：用正的比值求参考角，再放进对应象限。

$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}$$

🔍 Watch out for

当心

1. Wrong sign on the missing coordinate
   > 缺的坐标符号取错
2. Stopping at the calculator angle
   > 停在计算器给的角上
3. Mixing up x and y in the ratios
   > 比值里 x 和 y 用反了
4. Reading the sign clue as the wrong quadrant
   > 把符号线索读成了错的象限
5. Giving r a negative sign
   > 给 r 加了负号

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can use SOH CAH TOA and Pythagoras, and name the quadrant of a point.
  > 我会用 SOH CAH TOA 和勾股定理，也能说出一个点在哪个象限。

  (5′)
- ○ I can turn "one ratio plus a sign clue" into a quadrant and a labelled point.
  > 我能把「一个比值＋一个符号线索」变成一个象限和一个标好坐标的点。

  (5′)
- ○ I can follow, then finish, a point–triangle–ratios–angle solution.
  > 我能看懂并补全「定点—画三角形—写比值—求角」的解法。

  (8′)
- ○ I can find all three ratios and θ on my own with at most two hints.
  > 我能在最多两个提示下独立求出三个比值和 θ。

  (7′)
- ○ I can spot the five classic sign and quadrant mistakes.
  > 我能找出五个经典的符号与象限错误。

  (7′)
- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。

  (8′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. A right triangle has legs 6 and 8. What is the sine of the angle opposite the side of length 6?

> 直角三角形两直角边为 6 和 8。长 6 的边所对角的正弦是多少?

A. $\frac{3}{4}$ B. $\frac{4}{5}$ C. $\frac{3}{5}$ D. $\frac{4}{3}$

If you missed it · 如果做错了 — **SOH CAH TOA and Pythagoras**

> **直角三角形：SOH CAH TOA 与勾股定理**
> : In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.
>
> 在直角三角形中，从角 θ 看：$\sin\theta = \frac{\text{对边}}{\text{斜边}}$，$\cos\theta = \frac{\text{邻边}}{\text{斜边}}$，$\tan\theta = \frac{\text{对边}}{\text{邻边}}$。斜边对着直角；$a^2 + b^2 = c^2$。计算器设为角度（DEG）模式。

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
   > 两直角边 3 和 4：斜边 $\sqrt{9+16} = 5$；对着 3 的角 $\sin\theta = \frac{3}{5}$。
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.
   > 求角：$\theta = \sin^{-1}(0.6) \approx 36.9°$。

2. Find $\sqrt{9^2 + 12^2}$.

> 求 $\sqrt{9^2 + 12^2}$。

A. 21 B. 15 C. $\sqrt{21}$ D. 225

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**
> : Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.
>
> 记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

3. Simplify $\dfrac{-6}{-8}$.

> 化简 $\dfrac{-6}{-8}$。

A. $\frac{3}{4}$ B. $-\frac{3}{4}$ C. $-\frac{4}{3}$ D. $\frac{4}{3}$

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

> 减去负数等于加上它：$5-(-3)=5+3=8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2=16$，但 $-4^2=-16$（先算平方）。

1. $-7-(-2)$: change "minus negative" to "plus": $-7+2=-5$.
   > $-7-(-2)$：把「减负」改成「加」：$-7+2=-5$。
2. $(-3)(-5)=15$ (same signs), $(-3)(5)=-15$ (different signs).
   > $(-3)(-5)=15$（同号），$(-3)(5)=-15$（异号）。
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.
   > 把 $x=-3$ 代入 x^2：先加括号，$(-3)^2=9$。

4. The point $(-4,3)$ is in which quadrant?

> 点 $(-4,3)$ 在第几象限?

A. I B. III C. IV D. II

If you missed it · 如果做错了 — **Reading a graph: points, intercepts, slope**

> **读图：点、截距、斜率**

: A point is (x,y): across first, then up. x-intercepts are where $y=0$ (the graph crosses the horizontal axis); the y-intercept is where $x=0$. Slope $=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}$.

> 点写作 (x,y)：先横后纵。x 截距是 $y=0$ 的地方（图象穿过横轴处）；y 截距是 $x=0$ 的地方。斜率 $=\frac{\text{纵向变化}}{\text{横向变化}}=\frac{y_2-y_1}{x_2-x_1}$。

1. For $y=x^2-4$: y-intercept at $x=0$ is -4; x-intercepts solve $x^2-4=0$: $x=\pm 2$.
   > $y=x^2-4$：令 $x=0$ 得 y 截距 -4；令 $y=0$ 解 $x^2-4=0$ 得 x 截距 $x=\pm 2$。
2. Above the x-axis means $y>0$; below means $y<0$.
   > 在 x 轴上方表示 $y>0$；下方表示 $y<0$。

Answers · 答案: 1: C · 2: B · 3: A · 4: D

🔍 Common-errors sheet

常见错误单

1. Wrong sign on the missing coordinate

缺的坐标符号取错

$\sin\theta = \frac{5}{13}$ and θ is in quadrant II. Find $\cos\theta$.

$\sin\theta = \frac{5}{13}$，$\theta$ 在第二象限。求 $\cos\theta$。

1 $y = 5,\ r = 13,\ x^2 = 169 - 25 = 144$

> $y = 5,\ r = 13,\ x^2 = 169 - 25 = 144$

2 $x = 12$

3 $\cos\theta = \frac{12}{13}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Stopping at the calculator angle

停在计算器给的角上

$P(8, -15)$ is on the terminal arm. Find θ, $0^\circ \le \theta < 360^\circ$, to the nearest tenth.

$P(8, -15)$ 在终边上。求 θ（$0^\circ \le \theta < 360^\circ$，精确到 0.1°）。

1 $\tan\theta = -\frac{15}{8}$

2 $\tan^{-1}\left(-\frac{15}{8}\right) \approx -61.9^\circ$

3 $\theta = -61.9^\circ$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Mixing up x and y in the ratios

比值里 x 和 y 用反了

$P(-9, 40)$ is on the terminal arm. Find $\sin\theta$.

$P(-9, 40)$ 在终边上。求 $\sin\theta$。

1 $r = \sqrt{81 + 1600} = 41$

2 $\sin\theta = \frac{x}{r} = -\frac{9}{41}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Reading the sign clue as the wrong quadrant

把符号线索读成了错的象限

$\sin\theta = -0.6$ and $\tan\theta > 0$. Find θ, $0^\circ \le \theta < 360^\circ$, to the nearest tenth.

$\sin\theta = -0.6$ 且 $\tan\theta > 0$。求 θ（$0^\circ \le \theta < 360^\circ$，精确到 0.1°）。

1 sin negative, so Q IV

sin 为负，所以在第四象限

2 Reference angle $\sin^{-1} 0.6 \approx 36.9^\circ$

参考角 $\sin^{-1} 0.6 \approx 36.9^\circ$

3 $\theta = 360^\circ - 36.9^\circ = 323.1^\circ$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Giving r a negative sign

给 r 加了负号

$P(-5, -12)$ is on the terminal arm. Find $\sin\theta$.

$P(-5, -12)$ 在终边上。求 $\sin\theta$。

1 Both coordinates negative, so $r = -13$

两个坐标都为负，所以 $r = -13$

2 $\sin\theta = \frac{-12}{-13} = \frac{12}{13}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** A square root from a calculator is always positive, so the quadrant never got a say. In Q II the point is left of the y-axis.

   > 计算器开方总给正数，象限根本没被考虑进去。第二象限的点在 y 轴左边。

   $x = \pm 12$; Q II needs $x < 0$, so $x = -12$ and $\cos\theta = -\frac{12}{13}$.

   > $x = \pm 12$；第二象限要 $x < 0$，所以 $x = -12$，$\cos\theta = -\frac{12}{13}$。

2. **line · 第 3 —** The calculator answers in its own range, not the range the question set. Its output is a clue, not the answer.

   > 计算器按自己的范围给答案，不按题目要求的范围。它给的只是线索，不是答案。

   Q IV, reference angle $\tan^{-1}\frac{15}{8} \approx 61.9°$: $\theta = 360° - 61.9° = 298.1°$.

   > 第四象限，参考角 $\tan^{-1}\frac{15}{8} \approx 61.9°$：$\theta = 360° - 61.9° = 298.1°$。

3. **line · 第 2 —** SOH CAH TOA was learned with "opposite" and "adjacent", and the switch to x and y did not stick. Sine goes with the height y.

   > SOH CAH TOA 是用「对边」「邻边」学的，换成 x、y 后没记牢。正弦对应的是高度 y。

   $\sin\theta = \frac{y}{r} = \frac{40}{41}$ (and $\cos\theta = \frac{x}{r} = -\frac{9}{41}$).

   > $\sin\theta = \frac{y}{r} = \frac{40}{41}$（而 $\cos\theta = \frac{x}{r} = -\frac{9}{41}$）。

4. **line · 第 1 —** Only one clue was used. sin negative allows Q III or Q IV; the second clue, tan positive, is what picks between them.

   > 只用了一条线索。sin 为负可能是第三或第四象限；第二条线索 tan 为正才能二选一。

   $\sin < 0$ and $\tan > 0$: Q III. $\theta = 180° + 36.9° = 216.9°$.

   > $\sin < 0$ 且 $\tan > 0$：第三象限。$\theta = 180° + 36.9° = 216.9°$。

5. **line · 第 1 —** It treats r like a coordinate. r is a distance from the origin, and a distance is never negative.

   > 把 r 当成了坐标。r 是到原点的距离，距离不会是负的。

   $r = \sqrt{25 + 144} = 13$, so $\sin\theta = -\frac{12}{13}$ (Q III, sin negative ✓).

   > $r = \sqrt{25 + 144} = 13$，所以 $\sin\theta = -\frac{12}{13}$（第三象限，sin 为负 ✓）。