The cosine law

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** Mark what you know on a sketch and write each side next to the angle across from it (a faces A). Two sides and the angle BETWEEN them is SAS; three sides is SSS.
2. 🎯 **Asked** A side (SAS: the side across from the known angle) or an angle (SSS)? Give sides to the nearest tenth and angles in degrees.
3. ✏️ **Picture it** A rough triangle drawn roughly to scale. The biggest angle faces the longest side, and an obtuse angle makes its opposite side longer than Pythagoras would give.
4. 🧰 **Tool** SAS: $c^2 = a^2 + b^2 - 2ab\cos C$; work out $2ab\cos C$ first, subtract, then square-root. SSS: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, then $\cos^{-1}$.
5. ✅ **Check** Calculator in DEGREE mode. Does the biggest angle face the longest side? Is the answer a length, not c^2?

⭐ Must know

- $c^2 = a^2 + b^2 - 2ab\cos C$ (c opposite C); $C = 90^\circ$ gives Pythagoras.
- SAS → third side; SSS → angle from $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.
- $\cos C < 0$ → C is obtuse; $\cos^{-1}$ handles it, no ambiguous case.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

🔍 Watch out for

1. Subtracting before multiplying
2. Stopping at c squared
3. Dropping the minus sign of an obtuse cosine
4. Subtracting the wrong side when finding an angle
5. Using the bearing as the angle in the triangle

🌱 Small wins — tick as you go

- ○ I can find $\cos 60^\circ$, multiply by a negative cosine, and solve $36 = 61 - 60\cos C$ for $\cos C$. (5′)
- ○ I can tell SAS from SSS and name the side that sits across from the known angle. (5′)
- ○ I can follow, then finish, a cosine-law solution for a missing side. (8′)
- ○ I can find the side across from an obtuse angle with at most two hints. (8′)
- ○ I can spot the five classic cosine-law mistakes in someone else's work. (8′)
- ○ I can do the Level 1–2 practice questions with my calculator in degree mode. (10′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. What is $\cos 60°$?
A. $\frac{1}{2}$ B. $\frac{\sqrt{3}}{2}$ C. 0 D. 1

If you missed it — **SOH CAH TOA and Pythagoras**: In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.

2. Evaluate $36 + 81 - 108 \times (-0.5)$.
A. 63 B. 171 C. -171 D. 58.5

If you missed it — **Negative numbers: add, subtract, multiply**: Subtracting a negative is adding: $5 - (-3) = 5 + 3 = 8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2 = 16$, but $-4^2 = -16$ (the square happens first).

1. $-7 - (-2)$: change "minus negative" to "plus": $-7 + 2 = -5$.
2. $(-3)(-5) = 15$ (same signs), $(-3)(5) = -15$ (different signs).
3. Plugging in $x = -3$ to x^2: write brackets first, $(-3)^2 = 9$.

3. $c^2 = 49$ and c is the length of a side. What is c?
A. 24.5 B. 7 C. -7 D. 2401

If you missed it — **Perfect squares and square roots**: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.

4. Solve $36 = 61 - 60\cos C$ for $\cos C$.
A. $\frac{5}{12}$ B. $-\frac{5}{12}$ C. $\frac{97}{60}$ D. 25

If you missed it — **Rearranging a formula**: Treat the other letters as numbers and undo operations in reverse order. $A = P(1+r)$ for r: divide by P, then subtract 1: $r = \frac{A}{P} - 1$.

1. $y = a(x-p)^2 + q$ for a: subtract q, divide by $(x-p)^2$: $a = \frac{y-q}{(x-p)^2}$.
2. Substitute numbers first if the question gives them — fewer letters, fewer slips.

Answers: 1: A · 2: B · 3: B · 4: A

🔍 Common-errors sheet

1. Subtracting before multiplying

In $\triangle ABC$, $a = 5$, $b = 9$, $\angle C = 40^\circ$. Find c.

1 $c^2 = 25 + 81 - 90\cos 40^\circ$

2 $c^2 = (106 - 90)\cos 40^\circ = 16(0.766) \approx 12.26$

3 $c \approx 3.5$

✍ Which line goes wrong first, and why?

2. Stopping at c squared

In $\triangle ABC$, $a = 6$, $b = 8$, $\angle C = 70^\circ$. Find c.

1 $c^2 = 36 + 64 - 96\cos 70^\circ$

2 $c^2 \approx 100 - 32.83 = 67.17$

3 $c \approx 67.2$

✍ Which line goes wrong first, and why?

3. Dropping the minus sign of an obtuse cosine

In $\triangle ABC$, $a = 6$, $b = 9$, $\angle C = 120^\circ$. Find c.

1 $c^2 = 36 + 81 - 108\cos 120^\circ$

2 $c^2 = 117 - 108(0.5) = 63$

3 $c \approx 7.9$

✍ Which line goes wrong first, and why?

4. Subtracting the wrong side when finding an angle

A triangle has sides $a = 5$, $b = 7$, $c = 9$. Find $\angle A$.

1 $\cos A = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{25 + 49 - 81}{70}$

2 $\cos A = -0.1$

3 $A \approx 95.7°$

✍ Which line goes wrong first, and why?

.....

5. Using the bearing as the angle in the triangle

A hiker walks 5 km due north, then turns onto a bearing of $060°$ and walks 4 km. How far is she from where she started?

1 The angle at the turn is $60°$.

2 $d^2 = 25 + 16 - 40\cos 60° = 21$

3 $d \approx 4.6$ km

✍ Which line goes wrong first, and why?

.....

Key (fold this part back)

1. **line 2 —** $90\cos 40°$ is one product. Multiplication comes before subtraction, so it must be worked out before it is taken away from 106. $90\cos 40° \approx 68.94$, so $c^2 \approx 106 - 68.94 = 37.06$ and $c \approx 6.1$.
2. **line 3 —** The formula gives c^2, not c. A side of 67 cannot sit across a $70°$ angle when the other two sides are only 6 and 8. Take the square root: $c = \sqrt{67.17} \approx 8.2$.
3. **line 2 —** $\cos 120° = -0.5$, not 0.5. Then $-108(-0.5) = +54$: for an obtuse angle the last term ADDS. $c^2 = 117 + 54 = 171$, so $c \approx 13.1$. Check: the side across from the obtuse angle must be the longest, and 7.9 is shorter than 9.
4. **line 1 —** That form finds the angle across from c, which is C. The side that is subtracted on top must be the side across from the angle you want. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{49 + 81 - 25}{126} = \dfrac{105}{126}$, so $A \approx 33.6°$. Check: $a = 5$ is the shortest side, so A must be the smallest angle.
5. **line 1 —** $60°$ is how far she TURNED. The angle inside the triangle is between the path back to the start (due south) and the new path: $180° - 60° = 120°$. $d^2 = 25 + 16 - 40\cos 120° = 41 + 20 = 61$, so $d \approx 7.8$ km.