The cosine law

余弦定理

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Mark what you know on a sketch and write each side next to the angle across from it (a faces A). Two sides and the angle BETWEEN them is SAS; three sides is SSS.

   > 在草图上标出已知量，把每条边写在它对面的角旁边（a 对着 A）。两边及其「夹角」是 SAS；三条边是 SSS。

2. 🎯 **Asked · 求什么** A side (SAS: the side across from the known angle) or an angle (SSS)? Give sides to the nearest tenth and angles in degrees.

   > 求边（SAS：已知角对面的那条边）还是求角（SSS）？边精确到 0.1，角用度。

3. ✏️ **Picture it · 画出来** A rough triangle drawn roughly to scale. The biggest angle faces the longest side, and an obtuse angle makes its opposite side longer than Pythagoras would give.

   > 大致按比例画个三角形。最大的角对最长的边；钝角对面的边比用勾股定理算出的还长。

4. 🧰 **Tool · 用什么工具** SAS: $c^2 = a^2 + b^2 - 2ab\cos C$; work out $2ab\cos C$ first, subtract, then square-root. SSS: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, then $\cos^{-1}$.

   > SAS：$c^2 = a^2 + b^2 - 2ab\cos C$；先算 $2ab\cos C$，再相减，最后开方。SSS：$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$，再用 $\cos^{-1}$。

5. ✅ **Check · 检验** Calculator in DEGREE mode. Does the biggest angle face the longest side? Is the answer a length, not c^2?

   > 计算器是「角度制」吗？最大的角对着最长的边吗？答案是边长，而不是 c^2 吗？

⭐ Must know

必须记住

- $c^2 = a^2 + b^2 - 2ab\cos C$ (c opposite C); $C = 90^\circ$ gives Pythagoras.
  > $c^2 = a^2 + b^2 - 2ab\cos C$（$c$ 对角 C）；$C = 90^\circ$ 时就是勾股定理。
- SAS → third side; SSS → angle from $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.
  > SAS → 求第三边；SSS → 由 $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ 求角。
- $\cos C < 0$ → C is obtuse; $\cos^{-1}$ handles it, no ambiguous case.
  > $\cos C < 0$ → C 为钝角；$\cos^{-1}$ 直接给出，没有两解问题。

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

🔍 Watch out for

当心

1. Subtracting before multiplying
   > 先减后乘
2. Stopping at c squared
   > 停在了 c 的平方
3. Dropping the minus sign of an obtuse cosine
   > 钝角余弦丢了负号
4. Subtracting the wrong side when finding an angle
   > 求角时减错了边
5. Using the bearing as the angle in the triangle
   > 把方位角当成三角形内角

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can find $\cos 60^\circ$, multiply by a negative cosine, and solve $36 = 61 - 60\cos C$ for $\cos C$.
  > 我会求 $\cos 60^\circ$，会乘负的余弦值，也会从 $36 = 61 - 60\cos C$ 解出 $\cos C$。
  (5′)
- ○ I can tell SAS from SSS and name the side that sits across from the known angle.
  > 我能分清 SAS 和 SSS，并说出已知角对面的那条边。
  (5′)
- ○ I can follow, then finish, a cosine-law solution for a missing side.
  > 我能看懂并补全一道用余弦定理求边的题。
  (8′)
- ○ I can find the side across from an obtuse angle with at most two hints.
  > 我能在最多两个提示下，求出钝角对面的边。
  (8′)
- ○ I can spot the five classic cosine-law mistakes in someone else's work.
  > 我能在别人的解答里找出余弦定理的五个经典错误。
  (8′)
- ○ I can do the Level 1–2 practice questions with my calculator in degree mode.
  > 我能把计算器调到角度制，做出 1–2 级练习题。
  (10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. What is $\cos 60°$?

> $\cos 60°$ 等于多少?

A. $\frac{1}{2}$ B. $\frac{\sqrt{3}}{2}$ C. 0 D. 1

If you missed it · 如果做错了 — **SOH CAH TOA and Pythagoras**

> **直角三角形：SOH CAH TOA 与勾股定理**

: In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.

> 在直角三角形中，从角 θ 看：$\sin\theta = \frac{\text{对边}}{\text{斜边}}$，$\cos\theta = \frac{\text{邻边}}{\text{斜边}}$，$\tan\theta = \frac{\text{对边}}{\text{邻边}}$。斜边对着直角；$a^2 + b^2 = c^2$。计算器设为角度（DEG）模式。

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
   > 两直角边 3 和 4：斜边 $\sqrt{9+16} = 5$；对着 3 的角 $\sin\theta = \frac{3}{5}$。
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.
   > 求角：$\theta = \sin^{-1}(0.6) \approx 36.9°$。

2. Evaluate $36 + 81 - 108 \times (-0.5)$.

> 计算 $36 + 81 - 108 \times (-0.5)$。

A. 63 B. 171 C. -171 D. 58.5

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5 - (-3) = 5 + 3 = 8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2 = 16$, but $-4^2 = -16$ (the square happens first).

> 减去负数等于加上它：$5 - (-3) = 5 + 3 = 8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2 = 16$，但 $-4^2 = -16$（先算平方）。

1. $-7 - (-2)$: change "minus negative" to "plus": $-7 + 2 = -5$.
   > $-7 - (-2)$：把「减负」改成「加」：$-7 + 2 = -5$。
2. $(-3)(-5) = 15$ (same signs), $(-3)(5) = -15$ (different signs).
   > $(-3)(-5) = 15$（同号），$(-3)(5) = -15$（异号）。
3. Plugging in $x = -3$ to x^2: write brackets first, $(-3)^2 = 9$.
   > 把 $x = -3$ 代入 x^2：先加括号，$(-3)^2 = 9$。

3. $c^2 = 49$ and c is the length of a side. What is c?

> $c^2 = 49$，c 是一条边的长度。c 等于多少?

A. 24.5 B. 7 C. -7 D. 2401

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

4. Solve $36 = 61 - 60\cos C$ for $\cos C$.

> 由 $36 = 61 - 60\cos C$ 解出 $\cos C$。

A. $\frac{5}{12}$ B. $-\frac{5}{12}$ C. $\frac{97}{60}$ D. 25

If you missed it · 如果做错了 — **Rearranging a formula**

> **公式变形（求某个字母）**

: Treat the other letters as numbers and undo operations in reverse order. $A = P(1+r)$ for r: divide by P, then subtract 1: $r = \frac{A}{P} - 1$.

> 把其他字母当成数字，按相反顺序撤销运算。由 $A = P(1+r)$ 求 r：先除以 P，再减 1：$r = \frac{A}{P} - 1$。

1. $y = a(x-p)^2 + q$ for a: subtract q, divide by $(x-p)^2$: $a = \frac{y-q}{(x-p)^2}$.
   > 由 $y = a(x-p)^2 + q$ 求 a：减 q，再除以 $(x-p)^2$：$a = \frac{y-q}{(x-p)^2}$。
2. Substitute numbers first if the question gives them — fewer letters, fewer slips.
   > 题目给了数字就先代入——字母越少，越不容易错。

Answers · 答案: 1: A · 2: B · 3: B · 4: A

🔍 Common-errors sheet

常见错误单

1. Subtracting before multiplying

> 先减后乘

In $\triangle ABC$, $a = 5$, $b = 9$, $\angle C = 40^\circ$. Find c.

> 在 $\triangle ABC$ 中，$a = 5$，$b = 9$，$\angle C = 40^\circ$。求 c。

1 $c^2 = 25 + 81 - 90\cos 40^\circ$

2 $c^2 = (106 - 90)\cos 40^\circ = 16(0.766) \approx 12.26$

3 $c \approx 3.5$

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Stopping at c squared

> 停在了 c 的平方

In $\triangle ABC$, $a = 6$, $b = 8$, $\angle C = 70^\circ$. Find c.

> 在 $\triangle ABC$ 中，$a = 6$，$b = 8$，$\angle C = 70^\circ$。求 c。

1 $c^2 = 36 + 64 - 96\cos 70^\circ$

2 $c^2 \approx 100 - 32.83 = 67.17$

3 $c \approx 67.2$

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Dropping the minus sign of an obtuse cosine

钝角余弦丢了负号

In $\triangle ABC$, $a = 6$, $b = 9$, $\angle C = 120^\circ$. Find c.

在 $\triangle ABC$ 中，$a = 6$，$b = 9$，$\angle C = 120^\circ$。求 c。

1 $c^2 = 36 + 81 - 108\cos 120^\circ$

2 $c^2 = 117 - 108(0.5) = 63$

3 $c \approx 7.9$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Subtracting the wrong side when finding an angle

求角时减错了边

A triangle has sides $a = 5$, $b = 7$, $c = 9$. Find $\angle A$.

三角形三边 $a = 5$，$b = 7$，$c = 9$。求 $\angle A$。

1 $\cos A = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{25 + 49 - 81}{70}$

2 $\cos A = -0.1$

3 $A \approx 95.7^\circ$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Using the bearing as the angle in the triangle

> **把方位角当成三角形内角**

A hiker walks 5 km due north, then turns onto a bearing of 060° and walks 4 km. How far is she from where she started?

> 一名徒步者先向正北走 5 km，再转向方位角 060° 走 4 km。她离出发点多远?

1 The angle at the turn is 60°.

> 转弯处的角是 60°。

2 $d^2 = 25 + 16 - 40\cos 60^\circ = 21$

3 $d \approx 4.6$ km

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** $90\cos 40^\circ$ is one product. Multiplication comes before subtraction, so it must be worked out before it is taken away from 106.
   > $90\cos 40^\circ$ 是一个整体的乘积。先乘后减，要先把它算出来，再从 106 里减掉。
   $90\cos 40^\circ \approx 68.94$, so $c^2 \approx 106 - 68.94 = 37.06$ and $c \approx 6.1$.
   > $90\cos 40^\circ \approx 68.94$，所以 $c^2 \approx 106 - 68.94 = 37.06$，$c \approx 6.1$。
2. **line · 第 3 —** The formula gives c^2, not c. A side of 67 cannot sit across a 70° angle when the other two sides are only 6 and 8.
   > 公式算出的是 c^2，不是 c。另外两边只有 6 和 8，70° 角对面不可能是 67。
   Take the square root: $c = \sqrt{67.17} \approx 8.2$.
   > 开平方：$c = \sqrt{67.17} \approx 8.2$。
3. **line · 第 2 —** $\cos 120^\circ = -0.5$, not 0.5. Then $-108(-0.5) = +54$: for an obtuse angle the last term ADDS.
   > $\cos 120^\circ = -0.5$，不是 0.5。于是 $-108(-0.5) = +54$：钝角时最后一项是「加」。
   $c^2 = 117 + 54 = 171$, so $c \approx 13.1$. Check: the side across from the obtuse angle must be the longest, and 7.9 is shorter than 9.
   > $c^2 = 117 + 54 = 171$，$c \approx 13.1$。检验：钝角对面的边必须最长，而 7.9 比 9 还短。
4. **line · 第 1 —** That form finds the angle across from c, which is C. The side that is subtracted on top must be the side across from the angle you want.
   > 这个式子求的是 c 对面的角，也就是 C。分子里被减掉的那条边，必须是所求角对面的边。
   $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{49 + 81 - 25}{126} = \dfrac{105}{126}$, so $A \approx 33.6^\circ$. Check: $a = 5$ is the shortest side, so A must be the smallest angle.
   > $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{49 + 81 - 25}{126} = \dfrac{105}{126}$，所以 $A \approx 33.6^\circ$。检验：$a = 5$ 是最短边，A 应是最小角。
5. **line · 第 1 —** 60° is how far she TURNED. The angle inside the triangle is between the path back to the start (due south) and the new path: $180^\circ - 60^\circ = 120^\circ$.
   > 60° 是她「转过」的角度。三角形内角夹在回到起点的方向（正南）和新路线之间：$180^\circ - 60^\circ = 120^\circ$。
   $d^2 = 25 + 16 - 40\cos 120^\circ = 41 + 20 = 61$, so $d \approx 7.8$ km.
   > $d^2 = 25 + 16 - 40\cos 120^\circ = 41 + 20 = 61$，所以 $d \approx 7.8$ km。