The ambiguous case (SSA)

两解情形（SSA）

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 Given · 已知 Label the known angle A, the side across from it a, and the other known side b (it touches A). Is A acute or obtuse?
   > 标出已知角 A、它的对边 a、另一条已知边 b（它挨着 A）。A 是锐角还是钝角?
2. 🎯 Asked · 求什么 How many triangles (0, 1 or 2)? If asked to solve, every missing angle and side for EACH triangle, rounded as stated.
   > 有几个三角形（0、1 或 2）？如果要求解，每个三角形的所有未知角和边都要求出，按要求取近似。
3. ✏️ Picture it · 画出来 Draw A with side b to C, drop the height h from C, then swing side a from C like a gate and count where it hits the base.
   > 画出角 A 和边 b 到 C，从 C 作高 h，再把边 a 从 C 像门一样摆下来，数它碰到底边几次。
4. 🧰 Tool · 用什么工具 $h = b\sin A$. Acute A: $a < h$ none, $a = h$ one (right), $h < a < b$ two, $a \geq b$ one. Then $\sin B = \dfrac{b\sin A}{a}$ and $B_2 = 180^\circ - B_1$.
   > $h = b\sin A$。A 为锐角时：$a < h$ 无解，$a = h$ 一解（直角），$h < a < b$ 两解，$a \geq b$ 一解。再用 $\sin B = \dfrac{b\sin A}{a}$，$B_2 = 180^\circ - B_1$。
5. ✅ Check · 检验 For each triangle: angles add to 180°, and the longest side faces the largest angle. B_2 only counts if $A + B_2 < 180^\circ$.
   > 每个三角形：三角之和 180°，最长边对最大角。只有 $A + B_2 < 180^\circ$ 时 B_2 才算数。

⭐ Must know

必须记住

- SSA ($\angle A$, a, b): height $h = b\sin A$.
  > SSA（$\angle A$、a、b）：高 $h = b\sin A$。
- Acute A: $a < h$ none; $a = h$ one; $h < a < b$ TWO; $a \geq b$ one.
  > A 为锐角：$a < h$ 无；$a = h$ 一个；$h < a < b$ 两个；$a \geq b$ 一个。
- Two triangles: $B_2 = 180^\circ - B_1$; finish each triangle separately.
  > 两个三角形：$B_2 = 180^\circ - B_1$；两个三角形分别解完。

$$h = b\sin A$$

$$B_2 = 180^\circ - B_1$$

🔍 Watch out for

当心

1. Stopping at the first angle
   > 只求了第一个角就停
2. Using the wrong side for the height
   > 求高时用错了边
3. Comparing with only one length
   > 只比较了一个长度
4. Sine law ratio upside down
   > 正弦定理比例写反了
5. Reusing the first triangle's third angle
   > 第二个三角形照搬了第一个的第三角

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can find a side with $\sin$ in a right triangle and rearrange the sine law for $\sin B$.
  > 我会在直角三角形里用 $\sin$ 求边，也会把正弦定理变形成求 $\sin B$ 的式子。

  (5′)
- ○ I can label A, a, b and decide 0, 1 or 2 triangles from $h = b\sin A$.
  > 我能标出 A、a、b，并用 $h = b\sin A$ 判断有 0、1 还是 2 个三角形。

  (6′)
- ○ I can follow, then finish, a two-triangle solution: h-test, B_1, B_2, then each C and c.
  > 我能看懂并补全两解情形的完整过程：h 检验、B_1、B_2，再分别求 C 和 c。

  (9′)
- ○ I can solve both triangles on my own with at most two hints.
  > 我能在最多两个提示下独立解出两个三角形。

  (8′)
- ○ I can spot the five classic SSA mistakes in someone else's work.
  > 我能在别人的解答里找出 SSA 的五个经典错误。

  (8′)
- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。

  (9′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. A right triangle has hypotenuse 10 cm and an angle of $30°$. How long is the side opposite the $30°$ angle?

> 直角三角形斜边 10 cm，有一个 $30°$ 角。$30°$ 角所对的边多长?

A. 8.7 cm B. 20 cm C. 5 cm D. 5.8 cm

If you missed it · 如果做错了 — **SOH CAH TOA and Pythagoras**

> **直角三角形：SOH CAH TOA 与勾股定理**
> : In a RIGHT triangle, from angle θ: $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. The hypotenuse is opposite the right angle; $a^2 + b^2 = c^2$. Calculator in DEGREE mode.
>
> 在直角三角形中，从角 θ 看：$\sin\theta = \frac{\text{对边}}{\text{斜边}}$，$\cos\theta = \frac{\text{邻边}}{\text{斜边}}$，$\tan\theta = \frac{\text{对边}}{\text{邻边}}$。斜边对着直角；$a^2 + b^2 = c^2$。计算器设为角度（DEG）模式。

1. Legs 3 and 4: hypotenuse $\sqrt{9+16} = 5$; $\sin\theta = \frac{3}{5}$ for the angle opposite 3.
   > 两直角边 3 和 4：斜边 $\sqrt{9+16} = 5$；对着 3 的角 $\sin\theta = \frac{3}{5}$。
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.
   > 求角：$\theta = \sin^{-1}(0.6) \approx 36.9°$。

2. $\sin x = 0.5$ and x is acute. Find x.

> $\sin x = 0.5$，x 是锐角。求 x。

A. $30°$ B. $60°$ C. $45°$ D. $0.5°$

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> 在直角三角形中，从角 θ 看：$\sin\theta = \frac{\text{对边}}{\text{斜边}}$，$\cos\theta = \frac{\text{邻边}}{\text{斜边}}$，$\tan\theta = \frac{\text{对边}}{\text{邻边}}$。斜边对着直角；$a^2 + b^2 = c^2$。计算器设为角度（DEG）模式。

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   > 两直角边 3 和 4：斜边 $\sqrt{9+16} = 5$；对着 3 的角 $\sin\theta = \frac{3}{5}$。
2. Find an angle: $\theta = \sin^{-1}(0.6) \approx 36.9°$.
   > 求角：$\theta = \sin^{-1}(0.6) \approx 36.9°$。

3. Solve $\frac{a}{\sin A} = \frac{b}{\sin B}$ for $\sin B$.

> 由 $\frac{a}{\sin A} = \frac{b}{\sin B}$ 解出 $\sin B$。

A. $\sin B = \frac{a \sin A}{b}$ B. $\sin B = \frac{b \sin A}{a}$ C. $\sin B = \frac{ab}{\sin A}$ D. $\sin B = \frac{a}{b \sin A}$

If you missed it · 如果做错了 — **Rearranging a formula**

> **公式变形（求某个字母）**

: Treat the other letters as numbers and undo operations in reverse order. $A = P(1 + r)$ for r: divide by P, then subtract 1: $r = \frac{A}{P} - 1$.

> 把其他字母当成数字，按相反顺序撤销运算。由 $A = P(1 + r)$ 求 r：先除以 P，再减 1：$r = \frac{A}{P} - 1$。

1. $y = a(x - p)^2 + q$ for a: subtract q, divide by $(x - p)^2$: $a = \frac{y - q}{(x - p)^2}$.
   > 由 $y = a(x - p)^2 + q$ 求 a：减 q，再除以 $(x - p)^2$：$a = \frac{y - q}{(x - p)^2}$。
2. Substitute numbers first if the question gives them — fewer letters, fewer slips.
   > 题目给了数字就先代入——字母越少，越不容易错。

4. A triangle has angles $40°$ and $118°$. Find the third angle.

> 三角形有两个角是 $40°$ 和 $118°$，求第三个角。

A. $32°$ B. $202°$ C. $78°$ D. $22°$

If you missed it · 如果做错了 — **Solving a linear equation**

> **解一元一次方程**

: Do the same thing to both sides. Clear brackets, collect x on one side and numbers on the other, then divide. With fractions, multiply every term by the common denominator first.

> 两边做同样的事。先去括号，把含 x 的项移到一边、数字移到另一边，最后除以系数。有分数时先把每一项都乘以公分母。

1. $3(x - 2) = x + 8 \to 3x - 6 = x + 8 \to 2x = 14 \to x = 7$.
   > $3(x - 2) = x + 8 \to 3x - 6 = x + 8 \to 2x = 14 \to x = 7$。
2. $\frac{x}{2} + \frac{x}{3} = 5$: multiply every term by 6 → $3x + 2x = 30 \to x = 6$.
   > $\frac{x}{2} + \frac{x}{3} = 5$：每项乘 6 → $3x + 2x = 30 \to x = 6$。
3. Check by substituting: $3(7 - 2) = 15$ and $7 + 8 = 15$ ✓.
   > 代回检验：$3(7 - 2) = 15$，$7 + 8 = 15$ ✓。

Answers · 答案: 1: C · 2: A · 3: B · 4: D

🔍 Common-errors sheet

常见错误单

1. Stopping at the first angle

> 只求了第一个角就停

$\angle A = 38^\circ$, $b = 13$, $a = 10$. Find all possible values of $\angle B$ (nearest tenth).

> $\angle A = 38^\circ$，$b = 13$，$a = 10$。求 $\angle B$ 的所有可能值（精确到 0.1°）。

1 $\sin B = \dfrac{13 \sin 38^\circ}{10} \approx 0.8004$

2 $\sin^{-1} 0.8004 \approx 53.2^\circ$

3 So $\angle B \approx 53.2^\circ$.

> 所以 $\angle B \approx 53.2^\circ$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Using the wrong side for the height

> 求高时用错了边

$\angle A = 40^\circ$, $b = 15$, $a = 9$. How many triangles?

> $\angle A = 40^\circ$，$b = 15$，$a = 9$。有几个三角形?

1 $h = a \sin A = 9 \sin 40^\circ \approx 5.79$

2 $5.79 < 9 < 15$, so two triangles.

> $5.79 < 9 < 15$，所以有两个三角形。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Comparing with only one length

> 只比较了一个长度

$\angle A = 30^\circ$, $b = 20$, $a = 13$. How many triangles?

> $\angle A = 30^\circ$，$b = 20$，$a = 13$。有几个三角形?

1 $h = 20 \sin 30^\circ = 10$

2 $a = 13 > h$, so one triangle.

> $a = 13 > h$，所以一个三角形。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Sine law ratio upside down

> 正弦定理比例写反了

$\angle A = 45°$, $a = 12$, $b = 15$. Find all possible values of $\angle B$ (nearest tenth).

> $\angle A = 45°$，$a = 12$，$b = 15$。求 $\angle B$ 的所有可能值（精确到 0.1°）。

1 $\sin B = \dfrac{12 \sin 45°}{15} \approx 0.5657$

2 $B \approx 34.4°$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Reusing the first triangle's third angle

> 第二个三角形照搬了第一个的第三角

$\angle A = 36°$, $b = 16$, $a = 11$. There are two triangles. Find $\angle C$ in each (nearest tenth).

> $\angle A = 36°$，$b = 16$，$a = 11$，有两个三角形。分别求 $\angle C$（精确到 0.1°）。

1 $B_1 \approx 58.8°$, $C_1 = 180° - 36° - 58.8° = 85.2°$

> $B_1 \approx 58.8°$，$C_1 = 180° - 36° - 58.8° = 85.2°$

2 $B_2 = 180° - 58.8° = 121.2°$

3 $C_2 = 85.2°$, same as before

> $C_2 = 85.2°$，和前面一样

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 3 —** The calculator shows one angle, so it feels finished. But $\sin(180^\circ - 53.2^\circ)$ has the same value, and nothing here rules that angle out.

   > 计算器只显示一个角，感觉就做完了。但 $\sin(180^\circ - 53.2^\circ)$ 的值一样，题里没有任何条件排除它。

   $h = 13\sin 38^\circ \approx 8.00 < 10 < 13$: two triangles. $B_2 = 126.8^\circ$, and $38^\circ + 126.8^\circ < 180^\circ$ ✓.

   > $h = 13\sin 38^\circ \approx 8.00 < 10 < 13$：两个三角形。$B_2 = 126.8^\circ$，且 $38^\circ + 126.8^\circ < 180^\circ$ ✓。

2. **line · 第 1 —** "Angle A goes with side a" is right for the sine law but wrong for the height. The height drops from C, at the end of side b, so $h = b\sin A$.

   > 「角 A 配边 a」在正弦定理里对，求高时却不对。高是从 b 的端点 C 落下来的，所以 $h = b\sin A$。

   $h = 15\sin 40^\circ \approx 9.64 > 9$: side a is too short to reach the base. (Sine law agrees: $\sin B \approx 1.07 > 1$.)

   > $h = 15\sin 40^\circ \approx 9.64 > 9$：边 a 太短，够不到底边。（正弦定理也一致：$\sin B \approx 1.07 > 1$。）

3. **line · 第 2 —** "Longer than the height" only proves a reaches the base. Whether it reaches it once or twice depends on a compared with b.

   > 「比高长」只说明 a 能碰到底边。碰一次还是两次，要看 a 和 b 比。

   $10 < 13 < 20$: two triangles ($B \approx 50.3^\circ$ or 129.7°).

   > $10 < 13 < 20$：两个三角形（$B \approx 50.3^\circ$ 或 129.7°）。

4. **line · 第 1 —** The two fractions were cross-multiplied from memory instead of written out. Each side goes with the sine of its OWN opposite angle.

   > 凭印象交叉相乘，没先把比例写出来。每条边要配它自己对角的正弦。

   $\dfrac{\sin B}{15} = \dfrac{\sin 45^\circ}{12}$, so $\sin B \approx 0.8839$: $B \approx 62.1^\circ$ or 117.9° ($h \approx 10.61 < 12 < 15$).

   > $\dfrac{\sin B}{15} = \dfrac{\sin 45^\circ}{12}$，所以 $\sin B \approx 0.8839$：$B \approx 62.1^\circ$ 或 117.9°（$h \approx 10.61 < 12 < 15$）。

5. **line · 第 3 —** Only A, a and b are shared. Once B changes, the third angle and side change too: these are two different triangles.

   > 两个三角形共有的只有 A、a、b。B 一变，第三个角和第三边都跟着变：这是两个不同的三角形。

   $C_2 = 180^\circ - 36^\circ - 121.2^\circ = 22.8^\circ$. Check: $36 + 121.2 + 22.8 = 180$ ✓.

   > $C_2 = 180^\circ - 36^\circ - 121.2^\circ = 22.8^\circ$。验算：$36 + 121.2 + 22.8 = 180$ ✓。