Quadratic inequalities (sign analysis)

二次不等式（符号分析）

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Is one side 0? If not, move everything over first. Note the sign ($<$, $\leq$, $>$, $\geq$) and whether the x^2 coefficient is positive or negative.
   > 有一边是 0 吗？不是就先全部移过去。记下不等号（$<$, $\leq$, $>$, $\geq$），以及 x^2 的系数是正还是负。
2. 🎯 **Asked · 求什么** Every x that makes it true. Usually one interval or two pieces joined by "or"; sometimes all real numbers, one point, or nothing.
   > 使它成立的所有 x。通常是一个区间，或用「或」连接的两段；有时是全体实数、一个点，或无解。
3. ✏️ **Picture it · 画出来** A tiny parabola through the zeros (up if $a > 0$, down if $a < 0$), or a number line with the zeros and $+/-$ in each region.
   > 过零点画个小抛物线（$a > 0$ 开口向上，$a < 0$ 向下），或者画一条数轴，标出零点和每一段的 $+/-$。
4. 🧰 **Tool · 用什么工具** Factor (or use the formula) to get the zeros. Opens up: < 0 BETWEEN the zeros, > 0 OUTSIDE. Opens down: the other way round.
   > 分解（或用公式）求零点。开口向上：< 0 在零点「之间」，> 0 在零点「两侧」。开口向下：反过来。
5. ✅ **Check · 检验** Test one number inside your answer and one outside in the ORIGINAL inequality. Square brackets only for $\leq$ or $\geq$.
   > 在「原」不等式里代一个答案内的数和一个答案外的数。只有 $\leq$、$\geq$ 才用方括号。

⭐ Must know

必须记住

- Everything to one side, find the zeros, then decide the sign between and outside them.
  > 移到一边，求零点，再判断零点之间和两侧的符号。
- Opens up: negative BETWEEN the zeros, positive OUTSIDE.
  > 开口向上：零点之间为负，两侧为正。
- $\leq$, $\geq$ include the zeros; outside answers are joined by "or" ($\cup$).
  > $\leq$, $\geq$ 包括零点；两侧的答案用「或」($\cup$) 连接。

$$a(x-r)(x-s) < 0,\ a > 0,\ r < s \ \Rightarrow\ r < x < s$$

$$a(x-r)(x-s) > 0,\ a > 0 \ \Rightarrow\ x < r \text{ or } x > s$$

🔍 Watch out for

当心

1. Picking the wrong region
   > 区间选反了
2. Square-rooting an inequality
   > 对不等式直接开方
3. Factoring before comparing with zero
   > 没和 0 比较就分解
4. Multiplying by negative one without flipping
   > 乘以负一却没变号
5. Wrong brackets at the zeros
   > 零点处括号用错

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can flip an inequality when I multiply by a negative, and factor $x^2 + bx + c$.
  > 我会在乘以负数时把不等号反向，也会分解 $x^2 + bx + c$。

(5′)

- ○ I can get any quadratic inequality into the form "expression compared with 0" and name its zeros.
  > 我能把任何二次不等式化成「式子和 0 比较」的形式，并写出零点。

(6′)

- ○ I can follow, then finish, a zeros–sketch–pick solution.
  > 我能看懂并补全「求零点—画草图—选区间」的解法。

(8′)

- ○ I can solve one on my own with at most two hints and write it in interval notation.
  > 我能在最多两个提示下独立解一道，并用区间记号写答案。

(8′)

- ○ I can spot the five classic mistakes in someone else's work.
  > 我能在别人的解答里找出五个经典错误。

(8′)

- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。

(10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Solve $-3x + 6 > 12$.

> 解 $-3x + 6 > 12$。

A. $x > -2$ B. $x < -2$ C. $x > 2$ D. $x < 6$

If you missed it · 如果做错了 — **Inequalities: when the sign flips**

> **不等式：什么时候变号**

: Solve like an equation, EXCEPT: multiplying or dividing both sides by a negative number flips the sign. $-2x > 6 \rightarrow x < -3$. Test a number to be sure: $x = -4$ gives $8 > 6$ ✓.

> 像解方程一样解，唯一例外：两边同乘或同除以负数，不等号要反向。$-2x > 6 \rightarrow x < -3$。代一个数确认：$x = -4$ 得 $8 > 6$ ✓。

1. $5 - 3x \leq 11 \rightarrow -3x \leq 6 \rightarrow x \geq -2$ (divided by -3: flip).
   > $5 - 3x \leq 11 \rightarrow -3x \leq 6 \rightarrow x \geq -2$（除以 -3：变号）。
2. Number line: closed dot for $\leq, \geq$; open dot for $<, >$.
   > 数轴：$\leq, \geq$ 用实心点；$<, >$ 用空心点。

2. Factor $x^2 - x - 20$.

> 分解 $x^2 - x - 20$。

A. $(x - 4)(x + 5)$ B. $(x - 10)(x + 2)$ C. $(x + 4)(x - 5)$ D. $(x - 4)(x - 5)$

If you missed it · 如果做错了 — **Factoring $x^2 + bx + c$ (product and sum)**

> **分解 $x^2 + bx + c$（积与和）**

: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: $x^2 - x - 12 \rightarrow -4$ and $3 \rightarrow (x - 4)(x + 3)$. Check by expanding.

> 先提公因式。再找两个数，相乘等于 c、相加等于 b：$x^2 - x - 12 \rightarrow -4$ 和 $3 \rightarrow (x - 4)(x + 3)$。展开检验。

1. $x^2 + 7x + 10$: $2 \times 5 = 10$, $2 + 5 = 7 \rightarrow (x + 2)(x + 5)$.
   > $x^2 + 7x + 10$：$2 \times 5 = 10$，$2 + 5 = 7 \rightarrow (x + 2)(x + 5)$。
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.
   > 若 $c < 0$，两个数异号；若 $c > 0$，两个数与 b 同号。

3. Is $(x-3)(x+1)$ positive or negative when $x=0$?

> 当 $x=0$ 时，$(x-3)(x+1)$ 是正还是负?

A. Negative (-3) · 负（-3） B. Positive (3) · 正（3） C. Zero · 零 D. Positive (4) · 正（4）

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

> 减去负数等于加上它：$5-(-3)=5+3=8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2=16$，但 $-4^2=-16$（先算平方）。

1. $-7-(-2)$: change "minus negative" to "plus": $-7+2=-5$.
   > $-7-(-2)$：把「减负」改成「加」：$-7+2=-5$。
2. $(-3)(-5)=15$ (same signs), $(-3)(5)=-15$ (different signs).
   > $(-3)(-5)=15$（同号），$(-3)(5)=-15$（异号）。
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.
   > 把 $x=-3$ 代入 x^2：先加括号，$(-3)^2=9$。

4. Where does the graph of $y=(x+2)(x-6)$ cross the x-axis?

> $y=(x+2)(x-6)$ 的图象与 x 轴交于哪里?

A. $x=2$ and $x=-6$ B. $x=-12$ only C. $x=-2$ and $x=6$ D. $y=-12$

If you missed it · 如果做错了 — **Reading a graph: points, intercepts, slope**

> **读图：点、截距、斜率**

: A point is (x,y): across first, then up. x-intercepts are where $y=0$ (the graph crosses the horizontal axis); the y-intercept is where $x=0$. Slope $=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}$.

> 点写作 (x,y)：先横后纵。x 截距是 $y=0$ 的地方（图象穿过横轴处）；y 截距是 $x=0$ 的地方。斜率 $=\frac{\text{纵向变化}}{\text{横向变化}}=\frac{y_2-y_1}{x_2-x_1}$。

1. For $y=x^2-4$: y-intercept at $x=0$ is -4; x-intercepts solve $x^2-4=0$: $x=\pm 2$.
   > $y=x^2-4$：令 $x=0$ 得 y 截距 -4；令 $y=0$ 解 $x^2-4=0$ 得 x 截距 $x=\pm 2$。
2. Above the x-axis means $y>0$; below means $y<0$.
   > 在 x 轴上方表示 $y>0$；下方表示 $y<0$。

Answers · 答案: 1: B · 2: C · 3: A · 4: C

🔍 Common-errors sheet

常见错误单

1. Picking the wrong region

区间选反了

Solve $x^2 - 4x - 21 > 0$.

解 $x^2 - 4x - 21 > 0$。

1 $(x - 7)(x + 3) > 0$

2 Zeros -3 and 7

零点 -3 和 7

3 So $-3 < x < 7$

所以 $-3 < x < 7$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Square-rooting an inequality

对不等式直接开方

Solve $x^2 \geq 25$.

解 $x^2 \geq 25$。

1 Take square roots: $x \geq \pm 5$

两边开方：$x \geq \pm 5$

2 So $x \geq 5$ or $x \geq -5$

所以 $x \geq 5$ 或 $x \geq -5$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Factoring before comparing with zero

> **没和 0 比较就分解**

Solve $x^2 + 2x < 24$.

> 解 $x^2 + 2x < 24$。

1 $x(x + 2) < 24$

2 So $x < 24$ or $x + 2 < 24$

> 所以 $x < 24$ 或 $x + 2 < 24$

3 $x < 24$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Multiplying by negative one without flipping

> **乘以负一却没变号**

Solve $-x^2 + x + 12 > 0$.

> 解 $-x^2 + x + 12 > 0$。

1 Multiply by -1: $x^2 - x - 12 > 0$

> 乘以 -1：$x^2 - x - 12 > 0$

2 $(x - 4)(x + 3) > 0$

3 $x < -3$ or $x > 4$

> $x < -3$ 或 $x > 4$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Wrong brackets at the zeros

> **零点处括号用错**

Solve $x^2 - 2x - 8 \geq 0$.

> 解 $x^2 - 2x - 8 \geq 0$。

1 $(x-4)(x+2) \geq 0$, zeros -2 and 4

> $(x-4)(x+2) \geq 0$，零点 -2 和 4

2 Opens up, so outside

> 开口向上，取两侧

3 $(-\infty, -2) \cup (4, \infty)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 3 —** The answer "between the zeros" feels natural because the zeros are what you just found. But an upward parabola is ABOVE the axis outside its zeros.
   > 刚求出零点，「取中间」感觉最自然。但开口向上的抛物线在零点两侧才位于 x 轴上方。
   Sketch or test $x = 0$: $-21 > 0$ ×, so the middle is wrong. Answer: $x < -3$ or $x > 7$.
   > 画草图或代 $x = 0$：$-21 > 0$ ×，所以中间不对。答案：$x < -3$ 或 $x > 7$。
2. **line · 第 1 —** It copies the equation habit $x^2 = 25 \Rightarrow x = \pm 5$. With $\geq$, the -5 side runs the other way: $x = -10$ works but is not ≥ -5.
   > 照搬了方程的习惯 $x^2 = 25 \Rightarrow x = \pm 5$。换成 $\geq$ 时，-5 这一边方向相反：$x = -10$ 满足，却不 ≥ -5。
   $x^2 - 25 \geq 0$, $(x-5)(x+5) \geq 0$, opens up, outside: $x \leq -5$ or $x \geq 5$.
   > $x^2 - 25 \geq 0$，$(x-5)(x+5) \geq 0$，开口向上，取两侧：$x \leq -5$ 或 $x \geq 5$。
3. **line · 第 2 —** The sign rules for a product only work against 0. Knowing $x(x+2)$ is below 24 says nothing about each factor.
   > 乘积的符号规律只在和 0 比较时才成立。知道 $x(x+2) < 24$，并不能推出每个因式的情况。
   $x^2 + 2x - 24 < 0$, $(x+6)(x-4) < 0$, opens up, between: $-6 < x < 4$.
   > $x^2 + 2x - 24 < 0$，$(x+6)(x-4) < 0$，开口向上，取中间：$-6 < x < 4$。
4. **line · 第 1 —** Making a positive is a good move, but it changes the direction of every inequality, the same rule as $-2x > 6 \Rightarrow x < -3$.
   > 把 a 变正是好办法，但乘负数会让不等号反向，和 $-2x > 6 \Rightarrow x < -3$ 是同一条规则。
   $x^2 - x - 12 < 0$, so between the zeros: $-3 < x < 4$. (Or keep the original: it opens down and is positive between its zeros.)
   > $x^2 - x - 12 < 0$，取零点之间：$-3 < x < 4$。（或者不乘：原式开口向下，在零点之间为正。）
5. **line · 第 3 —** The zeros are the points where the expression equals 0, and $\geq$ asks for "0 or more", so they belong in the answer.
   > 零点处式子正好等于 0，而 $\geq$ 要的是「大于或等于 0」，所以零点要包含。
   Square brackets at both zeros: $(-\infty, -2] \cup [4, \infty)$. Check $x = 4$: $16 - 8 - 8 = 0 \geq 0$ ✓.
   > 两个零点都用方括号：$(-\infty, -2] \cup [4, \infty)$。验算 $x = 4$：$16 - 8 - 8 = 0 \geq 0$ ✓。