The discriminant

判别式

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Put the equation in the form $ax^2 + bx + c = 0$ (everything on one side), then write a, b, c with their signs. If there is a k, note which letter it lives in.
   > 先化成 $ax^2 + bx + c = 0$（全部移到一边），再写出带符号的 a、b、c。若有 k，看清它在哪个字母里。
2. 🎯 **Asked · 求什么** "How many roots" or "which k"? Equal roots → $D = 0$; two different → $D > 0$; none → $D < 0$; "real roots" (at least one) → $D \geq 0$.
   > 问的是「几个根」还是「k 取什么」？两等根 → $D = 0$；两不等根 → $D > 0$；无实根 → $D < 0$；「有实数根」（至少一个）→ $D \geq 0$。
3. ✏️ **Picture it · 画出来** The parabola and the x-axis: it crosses twice ($D > 0$), just touches at the vertex ($D = 0$), or misses ($D < 0$).
   > 抛物线和 x 轴：交两次（$D > 0$）、顶点刚好碰到（$D = 0$）、碰不到（$D < 0$）。
4. 🧰 **Tool · 用什么工具** $D = b^2 - 4ac$, with brackets around a negative b. With k, write D in terms of k, then solve the equation or inequality.
   > $D = b^2 - 4ac$，负的 b 要加括号。含 k 时，先用 k 写出 D，再解方程或不等式。
5. ✅ **Check · 检验** Put your k back: $D = 0$ should give a perfect square. For an inequality, test one k on each side. If k multiplies x^2, make sure $k \neq 0$.
   > 把 k 代回去：$D = 0$ 时应得到完全平方式。不等式就在分界点两边各试一个 k。若 k 是 x^2 的系数，要保证 $k \neq 0$。

⭐ Must know

必须记住

- $D = b^2 - 4ac$: $D > 0$ two roots, $D = 0$ one (double) root, $D < 0$ no real roots.
  > $D = b^2 - 4ac$：$D > 0$ 两个根，$D = 0$ 一个（重）根，$D < 0$ 无实数根。
- D a perfect square → rational roots → it factors.
  > D 是完全平方数 → 有理根 → 能因式分解。
- Unknown k: write D in k, set $= 0$, > 0 or < 0, solve.
  > 含 k：用 k 写出 D，令 $= 0$、> 0 或 < 0，解出 k。

$$D = b^2 - 4ac$$

🔍 Watch out for

当心

1. Calling a zero discriminant no solution
   > 把判别式为零当成无解
2. Losing the sign when squaring a negative b
   > 负的 b 平方时丢了符号
3. Using only part of the bracket as b
   > 只把括号的一部分当成 b
4. Forgetting that a cannot be zero
   > 忘了 a 不能为零
5. Not flipping the inequality for a negative
   > 除以负数时没有变号

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can square a negative number, solve $k^2 = 81$, and flip an inequality when I divide by a negative.
  > 我会算负数的平方，会解 $k^2 = 81$，除以负数时会把不等号反过来。

  (5′)
- ○ I can ask the five starting questions for a discriminant problem.
  > 我能对判别式题问出五个入手问题。

  (5′)
- ○ I can follow, then finish, a "write D in k, set $D = 0$, check" solution.
  > 我能看懂并补全「用 k 写出 D、令 $D = 0$、再检验」的解法。

  (8′)
- ○ I can find k on my own when k sits in front of x^2, with at most two hints.
  > 当 k 在 x^2 前面时，我能在最多两个提示下独立求出 k。

  (7′)
- ○ I can spot the five classic discriminant mistakes.
  > 我能找出判别式的五个经典错误。

  (8′)
- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。

  (10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Evaluate $b^2 - 4ac$ when $a = 1$, $b = -4$, $c = -5$.

> 当 $a = 1$，$b = -4$，$c = -5$ 时，求 $b^2 - 4ac$。

A. -4 B. 4 C. 36 D. -36

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5 - (-3) = 5 + 3 = 8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2 = 16$, but $-4^2 = -16$ (the square happens first).

> 减去负数等于加上它：$5 - (-3) = 5 + 3 = 8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2 = 16$，但 $-4^2 = -16$（先算平方）。

1. $-7 - (-2)$: change "minus negative" to "plus": $-7 + 2 = -5$.
   > $-7 - (-2)$：把「减负」改成「加」：$-7 + 2 = -5$。
2. $(-3)(-5) = 15$ (same signs), $(-3)(5) = -15$ (different signs).
   > $(-3)(-5) = 15$（同号），$(-3)(5) = -15$（异号）。
3. Plugging in $x = -3$ to x^2: write brackets first, $(-3)^2 = 9$.
   > 把 $x = -3$ 代入 x^2：先加括号，$(-3)^2 = 9$。

2. What is $(-7)^2$?

> $(-7)^2$ 等于多少?

A. -49 B. 49 C. 14 D. -14

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

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   > $(-3)(-5) = 15$（同号），$(-3)(5) = -15$（异号）。
3. Plugging in $x = -3$ to x^2: write brackets first, $(-3)^2 = 9$.
   > 把 $x = -3$ 代入 x^2：先加括号，$(-3)^2 = 9$。

3. Solve $k^2 = 81$.

解 $k^2 = 81$。

A. $k = \pm 9$ B. $k = 9$ C. $k = \pm 3$ D. $k = 40.5$

If you missed it · 如果做错了 — **Perfect squares and square roots**

完全平方数与平方根

: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.

   $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.

   $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

4. Solve $-3k > 12$.

解 $-3k > 12$。

A. $k > -4$ B. $k > 4$ C. $k < 4$ D. $k < -4$

If you missed it · 如果做错了 — **Inequalities: when the sign flips**

不等式：什么时候变号

: Solve like an equation, EXCEPT: multiplying or dividing both sides by a negative number flips the sign. $-2x > 6 \to x < -3$. Test a number to be sure: $x = -4$ gives $8 > 6$ ✓.

像解方程一样解，唯一例外：两边同乘或同除以负数，不等号要反向。$-2x > 6 \to x < -3$。代一个数确认：$x = -4$ 得 $8 > 6$ ✓。

1. $5 - 3x \leq 11 \to -3x \leq 6 \to x \geq -2$ (divided by -3: flip).

   $5 - 3x \leq 11 \to -3x \leq 6 \to x \geq -2$（除以 -3：变号）。
2. Number line: closed dot for $\leq, \geq$; open dot for $<, >$.

   数轴：$\leq, \geq$ 用实心点；$<, >$ 用空心点。

Answers · 答案: 1: C · 2: B · 3: A · 4: D

🔍 Common-errors sheet

常见错误单

1. Calling a zero discriminant no solution

> **把判别式为零当成无解**

State the nature of the roots of $x^2 - 10x + 25 = 0$.

> 判断 $x^2 - 10x + 25 = 0$ 的根的情况。

1 $D = (-10)^2 - 4(1)(25) = 100 - 100 = 0$

2 $D = 0$, so there are no real roots.

> $D = 0$，所以没有实数根。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Losing the sign when squaring a negative b

> **负的 b 平方时丢了符号**

State the nature of the roots of $2x^2 - 6x + 3 = 0$.

> 判断 $2x^2 - 6x + 3 = 0$ 的根的情况。

1 $D = -6^2 - 4(2)(3) = -36 - 24 = -60$

2 $D < 0$, so no real roots.

> $D < 0$，所以没有实数根。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Using only part of the bracket as b

> **只把括号的一部分当成 b**

Find k so that $x^2 + (k + 1)x + 9 = 0$ has two equal real roots.

> 求 k，使 $x^2 + (k + 1)x + 9 = 0$ 有两个相等的实数根。

1 $b = k$, so $D = k^2 - 36$

> $b = k$，所以 $D = k^2 - 36$

2 $k^2 - 36 = 0$, so $k = \pm 6$.

> $k^2 - 36 = 0$，所以 $k = \pm 6$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Forgetting that a cannot be zero

> **忘了 a 不能为零**

For which k does $kx^2 + 4x + 1 = 0$ have two different real roots?

> k 取何值时，$kx^2 + 4x + 1 = 0$ 有两个不相等的实数根?

1 $D = 16 - 4k$

2 $16 - 4k > 0$, so $k < 4$

> $16 - 4k > 0$，所以 $k < 4$

3 Answer: $k < 4$.

> 答案：$k < 4$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

5. Not flipping the inequality for a negative

> **除以负数时没有变号**

For which k does $x^2 + 6x + k = 0$ have no real roots?

> k 取何值时，$x^2 + 6x + k = 0$ 没有实数根?

1 $D = 36 - 4k < 0$

2 $-4k < -36$

3 $k < 9$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** "Zero" sounds like "nothing". But $\sqrt{0} = 0$, so $\pm\sqrt{D}$ gives the same number twice: one repeated root.

   > 「零」听起来像「没有」。但 $\sqrt{0} = 0$，所以 $\pm\sqrt{D}$ 两次得到同一个数：一个重根。

   $D = 0$: two equal real roots. In fact $(x-5)^2 = 0$, so $x = 5$.

   > $D = 0$：两个相等的实数根。也就是 $(x-5)^2 = 0$，$x = 5$。

2. **line · 第 1 —** Without brackets, -6^2 means $-(6^2) = -36$. The whole $b = -6$ must be squared: $(-6)^2 = 36$. A square is never negative.

   > 不加括号时 -6^2 表示 $-(6^2) = -36$。要把整个 $b = -6$ 平方：$(-6)^2 = 36$。平方永远不会是负数。

   $D = 36 - 24 = 12 > 0$: two different real roots (irrational, since 12 is not a perfect square).

   > $D = 36 - 24 = 12 > 0$：两个不相等的实数根（是无理数，因为 12 不是完全平方数）。

3. **line · 第 1 —** b is everything in front of x, so $b = k + 1$. Squaring only k gives a different equation.

   > b 是 x 前面的全部，所以 $b = k + 1$。只平方 k 就成了另一个方程。

   $D = (k+1)^2 - 36 = 0 \Rightarrow k + 1 = \pm 6 \Rightarrow k = 5$ or $k = -7$.

   > $D = (k+1)^2 - 36 = 0 \Rightarrow k + 1 = \pm 6 \Rightarrow k = 5$ 或 $k = -7$。

4. **line · 第 3 —** $k = 0$ is less than 4, but then the equation is $4x + 1 = 0$, which is linear and has only one root.

   > $k = 0$ 也小于 4，但这时方程是 $4x + 1 = 0$，是一次方程，只有一个根。

   Keep $D > 0$ and add $k \neq 0$: $k < 4$ and $k \neq 0$.

   > 保留 $D > 0$，再加上 $k \neq 0$：$k < 4$ 且 $k \neq 0$。

5. **line · 第 3 —** Dividing both sides by -4 reverses the inequality. A quick test shows it: $k = 0$ gives $D = 36 > 0$, two roots, so $k < 9$ cannot be right.

   > 两边除以 -4，不等号要反向。试一下就知道：$k = 0$ 时 $D = 36 > 0$，有两个根，所以 $k < 9$ 不可能对。

   $-4k < -36 \Rightarrow k > 9$. Test $k = 10$: $D = 36 - 40 = -4 < 0$ ✓.

   > $-4k < -36 \Rightarrow k > 9$。试 $k = 10$：$D = 36 - 40 = -4 < 0$ ✓。