Completing the square

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** Read off a, b, c. Is $a = 1$? Is b even or odd? Those two facts tell you whether fractions are coming.
2. 🎯 **Asked** Every real x that solves the equation, exact (with a root) unless decimals are asked. There may be two, one or none.
3. ✏️ **Picture it** The parabola $y = ax^2 + bx + c$. Completing the square finds its axis $x = p$; the roots sit the same distance either side of it.
4. 🧰 **Tool** Divide by a, move c to the right, add $\left(\frac{b}{2}\right)^2$ to BOTH sides, write the left side as $\left(x + \frac{b}{2}\right)^2$, take $\pm\sqrt{}$.
5. ✅ **Check** Right side negative → no real roots, stop. Otherwise check with the quadratic formula, or put a decimal answer back in.

⭐ Must know

- Make $a = 1$ → move c → add $\left(\frac{b}{2}\right)^2$ to BOTH sides → take $\pm\sqrt{}$.
- Equation: add to both sides. Vertex form of a function: add AND subtract on one side.
- Right side negative after completing → no real roots; stop.

$$x^2 + bx = k \;\Rightarrow\; \left(x + \tfrac{b}{2}\right)^2 = k + \left(\tfrac{b}{2}\right)^2$$

🔍 Watch out for

1. Adding the square to one side only
2. Forgetting the plus-or-minus
3. Dividing only some terms by a
4. Wrong sign inside the bracket
5. Adding b squared instead of half of b squared

🌱 Small wins — tick as you go

- ○ I can expand $(x + a)^2$, square a half like $\frac{5}{2}$, and solve $x^2 = 49$ with $\pm$. (5′)
- ○ I can ask the five starting questions before completing the square. (5′)
- ○ I can follow, then finish, a move–add–square–root solution. (8′)
- ○ I can complete the square on my own when $a \neq 1$, with at most two hints. (8′)
- ○ I can spot the five classic completing-the-square mistakes. (8′)
- ○ I can do the Level 1–2 practice questions. (10′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. Expand $(x+5)^2$.
A. x^2+25 B. $x^2+10x+25$ C. $x^2+5x+25$ D. $x^2+10x+10$

If you missed it — **Multiplying binomials, $(a+b)^2$**: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.

2. Which number makes $x^2-12x+\square$ a perfect square?
A. 36 B. 6 C. 144 D. -36

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3. What is $\left(\frac{5}{2}\right)^2$?
A. $\frac{5}{4}$ B. $\frac{25}{2}$ C. $\frac{25}{4}$ D. $\frac{10}{4}$

If you missed it — **Fractions: add, subtract, multiply, divide**: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.

4. Solve $x^2=49$.
A. $x=7$ B. $x=24.5$ C. $x=\pm 49$ D. $x=\pm 7$

If you missed it — **Perfect squares and square roots**: Know the squares $1,4,9,\ldots,225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
2. $x^2=49$ has TWO answers $x=\pm 7$, but $\sqrt{49}=7$ only.

5. Simplify $\sqrt{18}$.
A. $9\sqrt{2}$ B. $3\sqrt{2}$ C. $2\sqrt{3}$ D. 6

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Answers: 1: B · 2: A · 3: C · 4: D · 5: B

🔍 Common-errors sheet

1. Adding the square to one side only

Solve $x^2 - 8x + 5 = 0$ by completing the square.

1 $x^2 - 8x = -5$

2 $x^2 - 8x + 16 = -5$

3 $(x - 4)^2 = -5$, so no real solution.

✍ Which line goes wrong first, and why?

2. Forgetting the plus-or-minus

Solve $x^2 + 4x - 21 = 0$ by completing the square.

1 $x^2 + 4x = 21$

2 $x^2 + 4x + 4 = 25$, so $(x + 2)^2 = 25$

3 $x + 2 = 5$, so $x = 3$.

✍ Which line goes wrong first, and why?

3. Dividing only some terms by a

Solve $3x^2 + 6x - 1 = 0$ by completing the square.

1 Divide the x terms by 3: $x^2 + 2x - 1 = 0$

2 $x^2 + 2x = 1$, so $(x + 1)^2 = 2$

3 $x = -1 \pm \sqrt{2}$

✍ Which line goes wrong first, and why?

4. Wrong sign inside the bracket

Solve $x^2 - 6x - 2 = 0$ by completing the square.

1 $x^2 - 6x = 2$

2 $x^2 - 6x + 9 = 11$

3 $(x + 3)^2 = 11$

4 $x = -3 \pm \sqrt{11}$

✍ Which line goes wrong first, and why?

5. Adding b squared instead of half of b squared

Solve $x^2 + 10x + 9 = 0$ by completing the square.

1 $x^2 + 10x = -9$

2 $x^2 + 10x + 100 = 91$

3 $(x + 10)^2 = 91$, so $x = -10 \pm \sqrt{91}$

✍ Which line goes wrong first, and why?

Key (fold this part back)

1. **line 2 —** An equation is a balance. Adding 16 to the left only changes the equation, and here it even turns two real roots into "none". $x^2 - 8x + 16 = -5 + 16 = 11$, so $(x - 4)^2 = 11$ and $x = 4 \pm \sqrt{11}$.
2. **line 3 —** Both 5 and -5 square to 25. Taking only the positive root loses half the answers. $x + 2 = \pm 5$, so $x = 3$ or $x = -7$.
3. **line 1 —** Dividing is legal only if EVERY term is divided, the constant too. Otherwise it is a different equation. $x^2 + 2x - \frac{1}{3} = 0$, $x^2 + 2x = \frac{1}{3}$, $(x + 1)^2 = \frac{4}{3}$, $x = -1 \pm \frac{2}{\sqrt{3}} = -1 \pm \frac{2\sqrt{3}}{3}$.
4. **line 3 —** The sign in the bracket is the sign of half of b. $(x + 3)^2$ expands to $x^2 + 6x + 9$, which is not what you had. Half of -6 is -3: $(x - 3)^2 = 11$, so $x = 3 \pm \sqrt{11}$.
5. **line 2 —** $(x + 10)^2$ has middle term $20x$, not $10x$. The number to add is $\left(\frac{10}{2}\right)^2 = 25$. $x^2 + 10x + 25 = 16$, $(x + 5)^2 = 16$, $x + 5 = \pm 4$, so $x = -1$ or $x = -9$.