Completing the square

配方法

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Read off a, b, c. Is $a = 1$? Is b even or odd? Those two facts tell you whether fractions are coming.
   > 读出 a、b、c。a 是不是 1？b 是奇数还是偶数？这两点决定会不会出现分数。
2. 🎯 **Asked · 求什么** Every real x that solves the equation, exact (with a root) unless decimals are asked. There may be two, one or none.
   > 求所有满足方程的实数 x，除非题目要小数，否则写精确值（带根号）。可能有两个、一个或没有。
3. ✏️ **Picture it · 画出来** The parabola $y = ax^2 + bx + c$. Completing the square finds its axis $x = p$; the roots sit the same distance either side of it.
   > 抛物线 $y = ax^2 + bx + c$。配方能找到它的对称轴 $x = p$，两个根在对称轴两侧，距离相等。
4. 🧰 **Tool · 用什么工具** Divide by a, move c to the right, add $\left(\frac{b}{2}\right)^2$ to BOTH sides, write the left side as $\left(x + \frac{b}{2}\right)^2$, take $\pm\sqrt{\ }$.
   > 先除以 a，把常数项移到右边，两边同时加 $\left(\frac{b}{2}\right)^2$，左边写成 $\left(x + \frac{b}{2}\right)^2$，再开方取 $\pm$。
5. ✅ **Check · 检验** Right side negative → no real roots, stop. Otherwise check with the quadratic formula, or put a decimal answer back in.
   > 右边为负 → 无实数根，停。否则用求根公式核对，或把近似值代回原方程。

⭐ Must know

必须记住

- Make $a = 1$ → move c → add $\left(\frac{b}{2}\right)^2$ to BOTH sides → take $\pm\sqrt{\ }$.
  > 化 $a = 1$ → 移常数 → 两边都加 $\left(\frac{b}{2}\right)^2$ → 开方取 $\pm$。
- Equation: add to both sides. Vertex form of a function: add AND subtract on one side.
  > 解方程：两边同加。化函数顶点式：同一边加了再减。
- Right side negative after completing → no real roots; stop.
  > 配方后右边为负 → 无实数根，可以停。

$$x^2 + bx = k \;\Rightarrow\; \left(x + \tfrac{b}{2}\right)^2 = k + \left(\tfrac{b}{2}\right)^2$$

🔍 Watch out for

当心

1. Adding the square to one side only
   > 只在一边加平方数
2. Forgetting the plus-or-minus
   > 开方时忘了正负号
3. Dividing only some terms by a
   > 除以 a 时漏了几项
4. Wrong sign inside the bracket
   > 括号里的符号写反
5. Adding b squared instead of half of b squared
   > 加了 b 的平方，而不是 b 一半的平方

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can expand $(x+a)^2$, square a half like $\frac{5}{2}$, and solve $x^2=49$ with $\pm$.
  > 我会展开 $(x+a)^2$，会算 $\frac{5}{2}$ 这样的一半的平方，会用 $\pm$ 解 $x^2=49$。
  (5′)
- ○ I can ask the five starting questions before completing the square.
  > 配方之前，我能先问出五个入手问题。
  (5′)
- ○ I can follow, then finish, a move–add–square–root solution.
  > 我能看懂并补全「移项—两边加—写成平方—开方」的解法。
  (8′)
- ○ I can complete the square on my own when $a\neq 1$, with at most two hints.
  > $a\neq 1$ 时，我能在最多两个提示下独立配方。
  (8′)
- ○ I can spot the five classic completing-the-square mistakes.
  > 我能找出配方法的五个经典错误。
  (8′)
- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。
  (10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Expand $(x+5)^2$.

展开 $(x+5)^2$。

A. x^2+25 B. $x^2+10x+25$ C. $x^2+5x+25$ D. $x^2+10x+10$

If you missed it · 如果做错了 — **Multiplying binomials,** $(a+b)^2$

二项式相乘与 $(a+b)^2$

: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

每一项乘每一项：$(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$。平方会有中间项：$(x+3)^2=x^2+6x+9$，绝不是 x^2+9。

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
   $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$。
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.
   捷径：$(a\pm b)^2=a^2\pm 2ab+b^2$——中间项=2×首项×末项。

2. Which number makes $x^2-12x+\square$ a perfect square?

$x^2-12x+\square$ 中填哪个数，能凑成完全平方?

A. 36 B. 6 C. 144 D. -36

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   捷径：$(a\pm b)^2=a^2\pm 2ab+b^2$——中间项=2×首项×末项。

3. What is $\left(\frac{5}{2}\right)^2$?

> $\left(\frac{5}{2}\right)^2$ 等于多少?

A. $\frac{5}{4}$ B. $\frac{25}{2}$ C. $\frac{25}{4}$ D. $\frac{10}{4}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
   > $\frac{2}{3}-\frac{1}{4}$：公分母 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$。
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
   > $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$。
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$。

4. Solve $x^2=49$.

> 解 $x^2=49$。

A. $x=7$ B. $x=24.5$ C. $x=\pm49$ D. $x=\pm7$

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1,4,9,\ldots,225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1,4,9,\ldots,225$ 这些平方数。$\sqrt{ab}=\sqrt{a}\sqrt{b}$，所以 $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$。但 $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$：$\sqrt{9+16}=5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2=49$ has TWO answers $x=\pm7$, but $\sqrt{49}=7$ only.
   > $x^2=49$ 有两个解 $x=\pm7$，但 $\sqrt{49}$ 只等于 7。

5. Simplify $\sqrt{18}$.

> 化简 $\sqrt{18}$。

A. $9\sqrt{2}$ B. $3\sqrt{2}$ C. $2\sqrt{3}$ D. 6

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1,4,9,\ldots,225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1,4,9,\ldots,225$ 这些平方数。$\sqrt{ab}=\sqrt{a}\sqrt{b}$，所以 $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$。但 $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$：$\sqrt{9+16}=5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2=49$ has TWO answers $x=\pm7$, but $\sqrt{49}=7$ only.
   > $x^2=49$ 有两个解 $x=\pm7$，但 $\sqrt{49}$ 只等于 7。

Answers · 答案: 1: B · 2: A · 3: C · 4: D · 5: B

🔍 Common-errors sheet

常见错误单

1. Adding the square to one side only

> **只在一边加平方数**

Solve $x^2 - 8x + 5 = 0$ by completing the square.

> 用配方法解 $x^2 - 8x + 5 = 0$。

1 $x^2 - 8x = -5$

2 $x^2 - 8x + 16 = -5$

3 $(x - 4)^2 = -5$, so no real solution.

> $(x - 4)^2 = -5$，所以无实数解。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Forgetting the plus-or-minus

> **开方时忘了正负号**

Solve $x^2 + 4x - 21 = 0$ by completing the square.

> 用配方法解 $x^2 + 4x - 21 = 0$。

1 $x^2 + 4x = 21$

2 $x^2 + 4x + 4 = 25$, so $(x + 2)^2 = 25$

> $x^2 + 4x + 4 = 25$，即 $(x + 2)^2 = 25$

3 $x + 2 = 5$, so $x = 3$.

> $x + 2 = 5$，所以 $x = 3$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Dividing only some terms by a

> **除以 a 时漏了几项**

Solve $3x^2 + 6x - 1 = 0$ by completing the square.

> 用配方法解 $3x^2 + 6x - 1 = 0$。

1 Divide the x terms by 3: $x^2 + 2x - 1 = 0$

> 把含 x 的项除以 3：$x^2 + 2x - 1 = 0$

2 $x^2 + 2x = 1$, so $(x + 1)^2 = 2$

> $x^2 + 2x = 1$，即 $(x + 1)^2 = 2$

3 $x = -1 \pm \sqrt{2}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Wrong sign inside the bracket

> **括号里的符号写反**

Solve $x^2 - 6x - 2 = 0$ by completing the square.

> 用配方法解 $x^2 - 6x - 2 = 0$。

1 $x^2 - 6x = 2$

2 $x^2 - 6x + 9 = 11$

3 $(x + 3)^2 = 11$

4 $x = -3 \pm \sqrt{11}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Adding b squared instead of half of b squared

> **加了 b 的平方，而不是 b 一半的平方**

Solve $x^2 + 10x + 9 = 0$ by completing the square.

> 用配方法解 $x^2 + 10x + 9 = 0$。

1 $x^2 + 10x = -9$

2 $x^2 + 10x + 100 = 91$

3 $(x + 10)^2 = 91$, so $x = -10 \pm \sqrt{91}$

> $(x + 10)^2 = 91$，所以 $x = -10 \pm \sqrt{91}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** An equation is a balance. Adding 16 to the left only changes the equation, and here it even turns two real roots into "none".

   > 方程就是一架天平。只在左边加 16 就改变了方程，这里甚至把两个实数根变成了「无解」。

   $x^2 - 8x + 16 = -5 + 16 = 11$, so $(x - 4)^2 = 11$ and $x = 4 \pm \sqrt{11}$.

   > $x^2 - 8x + 16 = -5 + 16 = 11$，所以 $(x - 4)^2 = 11$，$x = 4 \pm \sqrt{11}$。

2. **line · 第 3 —** Both 5 and -5 square to 25. Taking only the positive root loses half the answers.

   > 5 和 -5 的平方都是 25。只取正根，就丢了一半的答案。

   $x + 2 = \pm 5$, so $x = 3$ or $x = -7$.

   > $x + 2 = \pm 5$，所以 $x = 3$ 或 $x = -7$。

3. **line · 第 1 —** Dividing is legal only if EVERY term is divided, the constant too. Otherwise it is a different equation.

   > 只有「每一项」都除，包括常数项，才合法。否则就变成了另一个方程。

   $x^2 + 2x - \frac{1}{3} = 0$, $x^2 + 2x = \frac{1}{3}$, $(x + 1)^2 = \frac{4}{3}$, $x = -1 \pm \frac{2}{\sqrt{3}} = -1 \pm \frac{2\sqrt{3}}{3}$.

   > $x^2 + 2x - \frac{1}{3} = 0$，$x^2 + 2x = \frac{1}{3}$，$(x + 1)^2 = \frac{4}{3}$，$x = -1 \pm \frac{2}{\sqrt{3}} = -1 \pm \frac{2\sqrt{3}}{3}$。

4. **line · 第 3 —** The sign in the bracket is the sign of half of b. $(x + 3)^2$ expands to $x^2 + 6x + 9$, which is not what you had.

   > 括号里的符号就是 b 的一半的符号。$(x + 3)^2$ 展开是 $x^2 + 6x + 9$，和原来的式子不一样。

   Half of -6 is -3: $(x - 3)^2 = 11$, so $x = 3 \pm \sqrt{11}$.

   > -6 的一半是 -3：$(x - 3)^2 = 11$，所以 $x = 3 \pm \sqrt{11}$。

5. **line · 第 2 —** $(x + 10)^2$ has middle term $20x$, not $10x$. The number to add is $\left(\frac{10}{2}\right)^2 = 25$.

   > $(x + 10)^2$ 的中间项是 $20x$，不是 $10x$。应加的是 $\left(\frac{10}{2}\right)^2 = 25$。

   $x^2 + 10x + 25 = 16$, $(x + 5)^2 = 16$, $x + 5 = \pm 4$, so $x = -1$ or $x = -9$.

   > $x^2 + 10x + 25 = 16$，$(x + 5)^2 = 16$，$x + 5 = \pm 4$，所以 $x = -1$ 或 $x = -9$。