Vertex form & transformations

顶点式与图象变换

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Read a, p and q. Rewrite the bracket as $(x - p)$ so the sign of p is honest: $(x + 3)$ means $p = -3$.
   > 读出 a、p、q。把括号写成 $(x - p)$ 的样子，p 的符号就不会看错：$(x + 3)$ 表示 $p = -3$。
2. 🎯 **Asked · 求什么** Which one is wanted: the vertex, axis, max or min, range, the transformations, the number of x-intercepts, or the equation itself (find a)?
   > 要的是哪一样：顶点、对称轴、最值、值域、变换过程、x 截距个数，还是方程本身（求 a）？
3. ✏️ **Picture it · 画出来** Plot the vertex, draw the axis $x = p$ as a dashed line, then step out 1, 2, 3 units and go up a, $4a$, $9a$ on both sides.
   > 先描顶点，用虚线画对称轴 $x = p$，再从顶点横走 1、2、3，竖走 a、$4a$、$9a$，两边都画。
4. 🧰 **Tool · 用什么工具** To describe: a stretches or flips, p moves left or right, q moves up or down. To find the equation: write $y = a(x - p)^2 + q$ with the vertex filled in, substitute the OTHER point, solve for a.
   > 描述图象：a 管伸缩和翻折，p 管左右，q 管上下。求方程：把顶点填进 $y = a(x - p)^2 + q$，代入「另一个」点，解出 a。
5. ✅ **Check · 检验** Put $x = p$ into your equation: you must get q. Put the other point in: it must work. Sense check: a point below the vertex means $a < 0$.
   > 把 $x = p$ 代入方程，必须得到 q。把另一个点代入，也必须成立。合理性：点在顶点下方，a 就小于 0。

⭐ Must know

必须记住

- $y = a(x - p)^2 + q$: vertex (p, q), axis $x = p$. $(x + 3)^2$ means $p = -3$.
  > $y = a(x - p)^2 + q$：顶点 (p, q)，对称轴 $x = p$。$(x + 3)^2$ 表示 $p = -3$。
- $|a| > 1$ narrower, $|a| < 1$ wider, $a < 0$ flips. From the vertex: over 1, 2, 3 → up a, $4a$, $9a$.
  > $|a| > 1$ 变窄，$|a| < 1$ 变宽，$a < 0$ 翻转。从顶点出发：右移 1、2、3 → 上升 a、$4a$、$9a$。
- x-intercepts: a, q opposite signs → 2; $q = 0$ → 1; same signs → 0.
  > x 截距个数：a, q 异号 → 2 个；$q = 0$ → 1 个；同号 → 0 个。

$$y = a(x - p)^2 + q$$

$$(x,\ y) \ \rightarrow \ (x + p,\ ay + q)$$

🔍 Watch out for

当心

1. Wrong sign for p
   > p 的符号读反了
2. Taking only the positive square root
   > 开平方只取了正根
3. Substituting the vertex to find a
   > 求 a 时代入了顶点
4. Mixing up wider and narrower
   > 把变宽和变窄弄反了
5. Counting x-intercepts from q alone
   > 只看 q 判断 x 截距个数

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can substitute a negative number into a squared bracket, solve $4a - 3 = 5$, and solve $(x - 1)^2 = 9$.

> 我会把负数代入平方括号，会解 $4a - 3 = 5$，也会解 $(x - 1)^2 = 9$。

(5′)

○ I can ask the five starting questions for a parabola in vertex form.

> 我能对顶点式抛物线问出五个入手问题。

(5′)

○ I can follow, then finish, finding a from the vertex and one other point.

> 我能看懂并补全「由顶点和另一个点求 a」的过程。

(8′)

○ I can write a vertex-form equation on my own with at most two hints.

> 我能在最多两个提示下独立写出顶点式方程。

(7′)

○ I can spot the five classic mistakes in someone else's work.

> 我能在别人的解答里找出五个经典错误。

(8′)

○ I can do the Level 1–2 practice questions.

> 我能做出 1–2 级练习题。

(10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Evaluate $-2(x+3)^2+5$ when $x=-1$.

> 当 $x=-1$ 时，求 $-2(x+3)^2+5$ 的值。

A. -3 B. 13 C. 3 D. -11

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

> 减去负数等于加上它：$5-(-3)=5+3=8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2=16$，但 $-4^2=-16$（先算平方）。

1. $-7-(-2)$: change "minus negative" to "plus": $-7+2=-5$.
   > $-7-(-2)$：把「减负」改成「加」：$-7+2=-5$。
2. $(-3)(-5)=15$ (same signs), $(-3)(5)=-15$ (different signs).
   > $(-3)(-5)=15$（同号），$(-3)(5)=-15$（异号）。
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.
   > 把 $x=-3$ 代入 x^2：先加括号，$(-3)^2=9$。

2. Solve $5=4a-3$.

> 解 $5=4a-3$。

A. $a=2$ B. $a=\frac{1}{2}$ C. $a=8$ D. $a=-2$

If you missed it · 如果做错了 — **Solving a linear equation**

> **解一元一次方程**

: Do the same thing to both sides. Clear brackets, collect x on one side and numbers on the other, then divide. With fractions, multiply every term by the common denominator first.

> 两边做同样的事。先去括号，把含 x 的项移到一边、数字移到另一边，最后除以系数。有分数时先把每一项都乘以公分母。

1. $3(x-2)=x+8 \to 3x-6=x+8 \to 2x=14 \to x=7$.
   > $3(x-2)=x+8 \to 3x-6=x+8 \to 2x=14 \to x=7$。
2. $\frac{x}{2}+\frac{x}{3}=5$: multiply every term by 6 → $3x+2x=30 \to x=6$.
   > $\frac{x}{2}+\frac{x}{3}=5$：每项乘 6 → $3x+2x=30 \to x=6$。
3. Check by substituting: $3(7-2)=15$ and $7+8=15$ ✓.
   > 代回检验：$3(7-2)=15$，$7+8=15$ ✓。

3. Solve $(x-1)^2 = 9$.

> 解 $(x-1)^2 = 9$。

A. $x = 4$ only · 只有 $x = 4$ B. $x = 4$ or $x = -2$ C. $x = 10$ or $x = -8$ D. $x = 3$ or $x = -3$

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

4. Start at $(1, 2)$, move 4 units left and 3 units up. Where are you?

> 从 $(1, 2)$ 出发，向左 4 个单位、向上 3 个单位。到了哪里?

A. $(5, 5)$ B. $(-3, 5)$ C. $(-3, -1)$ D. $(5, -1)$

If you missed it · 如果做错了 — **Reading a graph: points, intercepts, slope**

> **读图：点、截距、斜率**

: A point is (x, y): across first, then up. x-intercepts are where $y = 0$ (the graph crosses the horizontal axis); the y-intercept is where $x = 0$. Slope $= \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

> 点写作 (x, y)：先横后纵。x 截距是 $y = 0$ 的地方（图象穿过横轴处）；y 截距是 $x = 0$ 的地方。斜率 $= \frac{\text{纵向变化}}{\text{横向变化}} = \frac{y_2 - y_1}{x_2 - x_1}$。

1. For $y = x^2 - 4$: y-intercept at $x = 0$ is -4; x-intercepts solve $x^2 - 4 = 0$: $x = \pm 2$.
   > $y = x^2 - 4$：令 $x = 0$ 得 y 截距 -4；令 $y = 0$ 解 $x^2 - 4 = 0$ 得 x 截距 $x = \pm 2$。
2. Above the x-axis means $y > 0$; below means $y < 0$.
   > 在 x 轴上方表示 $y > 0$；下方表示 $y < 0$。

Answers · 答案: 1: A · 2: A · 3: B · 4: B

🔍 Common-errors sheet

常见错误单

1. Wrong sign for p

> **p 的符号读反了**

State the vertex of $y = 3(x + 2)^2 - 7$.

> 写出 $y = 3(x + 2)^2 - 7$ 的顶点。

1 $p = 2$ and $q = -7$.

> $p = 2,\ q = -7$。

2 Vertex $(2, -7)$.

> 顶点 $(2, -7)$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Taking only the positive square root

> **开平方只取了正根**

Find the x-intercepts of $y = 2(x - 3)^2 - 8$.

> 求 $y = 2(x - 3)^2 - 8$ 的 x 截距。

1 Set $y = 0$: $2(x - 3)^2 = 8$.

> 令 $y = 0$：$2(x - 3)^2 = 8$。

2 $(x - 3)^2 = 4$.

> $(x - 3)^2 = 4$。

3 $x - 3 = 2$, so $x = 5$.

> $x - 3 = 2$，所以 $x = 5$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Substituting the vertex to find a

求 a 时代入了顶点

Find the equation of the parabola with vertex $(3, 2)$ through $(5, 10)$.

求顶点为 $(3, 2)$、经过 $(5, 10)$ 的抛物线方程。

1 $y = a(x-3)^2 + 2$.

$y = a(x-3)^2 + 2$。

2 Substitute $(3, 2)$: $2 = a(0)^2 + 2$.

代入 $(3, 2)$：$2 = a(0)^2 + 2$。

3 $2 = 2$ works for any a, so take $a = 1$: $y = (x-3)^2 + 2$.

$2 = 2$ 对任何 a 都成立，那就取 $a = 1$：$y = (x-3)^2 + 2$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Mixing up wider and narrower

把变宽和变窄弄反了

Describe how $y = \frac{1}{3}(x-1)^2 + 4$ is obtained from $y = x^2$.

说明 $y = \frac{1}{3}(x-1)^2 + 4$ 是怎样由 $y = x^2$ 得到的。

1 $a = \frac{1}{3}$, so the graph is stretched by a factor of 3 and gets narrower.

$a = \frac{1}{3}$，所以图象拉伸为 3 倍，变窄了。

2 Then it moves 1 right and 4 up.

然后向右 1、向上 4。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Counting x-intercepts from q alone

只看 q 判断 x 截距个数

How many x-intercepts does $y = -3(x+1)^2 - 2$ have?

$y = -3(x+1)^2 - 2$ 有几个 x 截距?

1 $q = -2$, so the vertex $(-1, -2)$ is below the x-axis.

$q = -2$，所以顶点 $(-1, -2)$ 在 x 轴下方。

2 A vertex below the axis means the graph crosses it twice: 2 intercepts.

顶点在 x 轴下方，图象就会穿过 x 轴两次：2 个截距。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 1 —** The form is $(x-p)$, so $(x+2)=(x-(-2))$ means $p=-2$. The bracket is zero at $x=-2$, and that is where the vertex sits.

   > 顶点式写的是 $(x-p)$，$(x+2)=(x-(-2))$，所以 $p=-2$。括号在 $x=-2$ 时等于零，顶点就在那里。

   $p=-2$, $q=-7$: vertex $(-2,-7)$.

   > $p=-2$，$q=-7$：顶点 $(-2,-7)$。

2. **line · 第 3 —** Both 2^2 and $(-2)^2$ equal 4. A parabola is symmetric, so an intercept at 5 (2 right of the axis $x=3$) has a partner 2 to the left.

   > 2^2 和 $(-2)^2$ 都等于 4。抛物线是对称的，5 在对称轴 $x=3$ 右边 2 处，左边 2 处一定还有一个截距。

   $x-3=\pm 2$, so $x=5$ or $x=1$.

   > $x-3=\pm 2$，所以 $x=5$ 或 $x=1$。

3. **line · 第 2 —** The vertex is already built into the equation, so it always fits and tells you nothing about a. Only the other point pins a down.

   > 顶点已经写进方程了，代进去永远成立，对求 a 毫无帮助。只有另一个点才能定出 a。

   Use $(5,10)$: $10=a(2)^2+2=4a+2$, so $a=2$ and $y=2(x-3)^2+2$.

   > 用 $(5,10)$：$10=a(2)^2+2=4a+2$，所以 $a=2$，$y=2(x-3)^2+2$。

4. **line · 第 1 —** The factor is a itself, not its flip. Every height is multiplied by $\frac{1}{3}$: from the vertex, over 3 goes up only $9\cdot\frac{1}{3}=3$ instead of 9, so the graph is flatter and wider.

   > 伸缩因子就是 a 本身，不是它的倒数。每个高度都乘以 $\frac{1}{3}$：从顶点横走 3，只上升 $9\cdot\frac{1}{3}=3$ 而不是 9，所以图象更平、更宽。

   Vertical stretch by a factor of $\frac{1}{3}$ (wider), then 1 right and 4 up. Vertex $(1,4)$.

   > 竖直方向伸缩为原来的 $\frac{1}{3}$（变宽），再向右 1、向上 4。顶点 $(1,4)$。

5. **line · 第 2 —** That is only true if the parabola opens UP. Here $a=-3<0$, so it opens down from a vertex that is already below the axis and never reaches it.

   > 只有开口向上时才这样。这里 $a=-3<0$，开口向下，顶点本来就在 x 轴下方，图象永远够不到 x 轴。

   a and q are both negative (same signs), so there are 0 x-intercepts. Check: $-3(x+1)^2=2$ needs a square equal to $-\frac{2}{3}$, which is impossible.

   > a 和 q 都是负数（同号），所以有 0 个 x 截距。检验：$-3(x+1)^2=2$ 要求平方等于 $-\frac{2}{3}$，不可能。