Standard form → vertex form

一般式化顶点式

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Write a, b, c with their signs. Note whether a is 1, another positive number, or negative.
   > 带符号写出 a、b、c。看清 a 是 1、别的正数，还是负数。
2. 🎯 **Asked · 求什么** The form $y = a(x-p)^2 + q$ with the SAME a, plus the vertex (p, q) and, if asked, the max/min value or the range.
   > 要的是 $y = a(x-p)^2 + q$，a 和原来「一样」；再写出顶点 (p, q)，题目要的话还有最值或值域。
3. ✏️ **Picture it · 画出来** Sketch where the vertex should be: $p = -\frac{b}{2a}$ sets the axis, and the sign of a says whether q is a minimum or a maximum.
   > 先估顶点在哪：$p = -\frac{b}{2a}$ 定对称轴，a 的符号决定 q 是最小值还是最大值。
4. 🧰 **Tool · 用什么工具** Factor a out of the x^2 and x terms only. Inside, add and subtract $\left(\frac{b'}{2}\right)^2$, where b' is the new x-coefficient. Move the subtracted number out, multiplied by a.
   > 只从 x^2 项和 x 项提出 a。括号里加上再减去 $\left(\frac{b'}{2}\right)^2$（$b'$ 是括号里新的 x 系数）。把减掉的数乘以 a 再移出括号。
5. ✅ **Check · 检验** Your p must equal $-\frac{b}{2a}$, and $f(p)$ in the ORIGINAL must equal your q. Expanding back is the full check.
   > 你的 p 必须等于 $-\frac{b}{2a}$，把 p 代入「原」式得到的值必须等于你的 q。展开回去是最彻底的检验。

⭐ Must know

必须记住

- $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$
- $a \neq 1$: factor a from the x^2 and x terms only; the subtracted square leaves the bracket times a.
  > $a \neq 1$：只从 x^2 项和 x 项提出 a；减掉的平方数移出括号时要乘 a。
- Check: vertex $x = -\frac{b}{2a}$, vertex $y = f\left(-\frac{b}{2a}\right)$.
  > 检验：顶点横坐标 $x = -\frac{b}{2a}$，纵坐标 $y = f\left(-\frac{b}{2a}\right)$。

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

$$y = ax^2 + bx + c \;\Rightarrow\; p = -\frac{b}{2a},\; q = f(p)$$

🔍 Watch out for

当心

1. Moving the number out without multiplying by a
   > 移出括号时没乘 a
2. Wrong sign inside the bracket
   > 括号里的符号写反了
3. Adding the square but never subtracting it
   > 只加了平方数，没有再减掉
4. Factoring out a negative a but keeping the old sign
   > 提出负的 a 却没给 x 项变号
5. Using b instead of half of b
   > 用了 b，而不是 b 的一半

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can expand $(x-4)^2$, spot a perfect square like $x^2+10x+25$, square a fraction, and distribute -2.

> 我会展开 $(x-4)^2$，能认出 $x^2+10x+25$ 这样的完全平方，会算分数的平方，会把 -2 乘进括号。

(5′)

○ I can say what vertex form will look like before I start, and work out $p=-\frac{b}{2a}$ as my target.

> 我能在动手前说出顶点式长什么样，并先算出 $p=-\frac{b}{2a}$ 作为目标。

(5′)

○ I can follow, then finish, the factor–add/subtract–move-out method when $a\neq 1$.

> 我能看懂并补全 $a\neq 1$ 时「提系数—加减平方—移出括号」的配方步骤。

(9′)

○ I can convert a quadratic with negative a on my own, with at most two hints, and give the maximum.

> 我能在最多两个提示下，独立把 a 为负的二次函数化成顶点式，并写出最大值。

(8′)

○ I can spot the five classic completing-the-square mistakes in someone else's work.

> 我能在别人的解答里找出配方的五个经典错误。

(8′)

○ I can do the Level 1–2 practice questions and check each vertex with $-\frac{b}{2a}$.

> 我能做出 1–2 级练习题，并用 $-\frac{b}{2a}$ 检验每个顶点。

(10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Expand $(x-4)^2$.

> 展开 $(x-4)^2$。

A. x^2-16 B. x^2+16 C. $x^2-8x+16$ D. $x^2-4x+16$

If you missed it · 如果做错了 — **Multiplying binomials, $(a+b)^2$**

> **二项式相乘与 $(a+b)^2$**

: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

> 每一项乘每一项：$(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$。平方会有中间项：$(x+3)^2=x^2+6x+9$，绝不是 x^2+9。

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
   > $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$。
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.
   > 捷径：$(a\pm b)^2=a^2\pm 2ab+b^2$——中间项＝2×首项×末项。

2. Which perfect square equals $x^2+10x+25$?

> $x^2+10x+25$ 等于哪个完全平方?

A. $(x+25)^2$ B. $(x+5)^2$ C. $(x+10)^2$ D. $(x-5)^2$

If you missed it · 如果做错了 — **Factoring x^2+bx+c (product and sum)**

> **分解 x^2+bx+c（积与和）**

: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: x^2-x-12 → -4 and 3 → $(x-4)(x+3)$. Check by expanding.

> 先提公因式。再找两个数，相乘等于 c、相加等于 b：x^2-x-12 → -4 和 3 → $(x-4)(x+3)$。展开检验。

1. $x^2+7x+10$: $2\times 5=10$, $2+5=7$ → $(x+2)(x+5)$.
   > $x^2+7x+10$：$2\times 5=10$，$2+5=7$ → $(x+2)(x+5)$。
2. If $c<0$ the two numbers have different signs; if $c>0$ they share the sign of b.
   > 若 $c<0$，两个数异号；若 $c>0$，两个数与 b 同号。

3. What is $\left(-\frac{3}{2}\right)^2$?

> $\left(-\frac{3}{2}\right)^2$ 等于多少?

A. $-\frac{9}{4}$ B. -3 C. $\frac{9}{2}$ D. $\frac{9}{4}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
   > $\frac{2}{3}-\frac{1}{4}$：公分母 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$。
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
   > $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$。
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$。

4. Expand $-2(x^2-4x)$.

> 去括号：$-2(x^2-4x)$。

A. $-2x^2+8x$ B. $-2x^2-8x$ C. $-2x^2-4x$ D. $2x^2-8x$

If you missed it · 如果做错了 — **Distributing (and the minus sign)**

> **去括号（尤其是负号）**

: Multiply EVERY term inside the bracket: $3(x-4)=3x-12$. A minus in front changes every sign inside: $-(x-5)=-x+5$, and $2-(x+3)=2-x-3=-x-1$.

> 括号里的每一项都要乘到：$3(x-4)=3x-12$。前面是负号时，括号里每一项都变号：$-(x-5)=-x+5$，$2-(x+3)=2-x-3=-x-1$。

1. $-2(3x-1)=-6x+2$ (the -1 becomes $+2$).
   > $-2(3x-1)=-6x+2$（-1 变成 $+2$）。
2. $x(x+2)-3(x+2)=x^2+2x-3x-6=x^2-x-6$.
   > $x(x+2)-3(x+2)=x^2+2x-3x-6=x^2-x-6$。

Answers · 答案: 1: C · 2: B · 3: D · 4: A

🔍 Common-errors sheet

常见错误单

1. Moving the number out without multiplying by a

> **移出括号时没乘 a**

Write $y = 2x^2 + 12x + 5$ in vertex form.

> 把 $y = 2x^2 + 12x + 5$ 化成顶点式。

1 $y = 2(x^2 + 6x + 9 - 9) + 5$

2 $y = 2(x + 3)^2 - 9 + 5$

3 $y = 2(x + 3)^2 - 4$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Wrong sign inside the bracket

> **括号里的符号写反了**

Write $y = x^2 + 8x + 3$ in vertex form and give the vertex.

> 把 $y = x^2 + 8x + 3$ 化成顶点式，并写出顶点。

1 Half of 8 is 4: $y = (x - 4)^2 - 16 + 3$

> 8 的一半是 4：$y = (x - 4)^2 - 16 + 3$

2 $y = (x - 4)^2 - 13$, vertex $(4, -13)$

> $y = (x - 4)^2 - 13$，顶点 $(4, -13)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Adding the square but never subtracting it

> **只加了平方数，没有再减掉**

Write $y = x^2 - 6x + 2$ in vertex form.

> 把 $y = x^2 - 6x + 2$ 化成顶点式。

1 Complete the square: $y = (x^2 - 6x + 9) + 2$

> 配方：$y = (x^2 - 6x + 9) + 2$

2 $y = (x - 3)^2 + 2$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Factoring out a negative a but keeping the old sign

> **提出负的 a 却没给 x 项变号**

Write $y = -x^2 + 6x + 1$ in vertex form.

> 把 $y = -x^2 + 6x + 1$ 化成顶点式。

1 $y = -(x^2 + 6x) + 1$

2 $y = -(x^2 + 6x + 9 - 9) + 1 = -(x + 3)^2 + 10$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Using b instead of half of b

> **用了 b，而不是 b 的一半**

Write $y = x^2 + 4x - 1$ in vertex form.

> 把 $y = x^2 + 4x - 1$ 化成顶点式。

1 $y = (x + 4)^2 - 16 - 1$

2 $y = (x + 4)^2 - 17$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** The -9 was inside a bracket that is multiplied by 2, so it is really worth -18. Lifting it out "as is" changes the value.

   > -9 原来在一个要乘 2 的括号里，它实际代表 -18。原样拿出来，式子的值就变了。

   $2(x+3)^2 - 18 + 5 = 2(x+3)^2 - 13$. Check: $f(-3) = 18 - 36 + 5 = -13$ ✓.

   > $2(x+3)^2 - 18 + 5 = 2(x+3)^2 - 13$。检验：$f(-3) = 18 - 36 + 5 = -13$ ✓。

2. **line · 第 1 —** Vertex form has a minus built in, $(x-p)^2$, so it feels like the bracket "should" have a minus. But $(x-4)^2$ expands to $x^2 - 8x + \ldots$, not $+8x$.

   > 顶点式自带减号 $(x-p)^2$，于是觉得括号里「应该」是减号。可 $(x-4)^2$ 展开是 $x^2 - 8x + \ldots$，不是 $+8x$。

   $y = (x+4)^2 - 16 + 3 = (x+4)^2 - 13$, vertex $(-4, -13)$. Check: $-\frac{8}{2} = -4$ ✓.

   > $y = (x+4)^2 - 16 + 3 = (x+4)^2 - 13$，顶点 $(-4, -13)$。检验：$-\frac{8}{2} = -4$ ✓。

3. **line · 第 1 —** Adding 9 is fine for an equation if you add it to both sides, but here there is only one side. Adding 9 alone makes a different parabola.

   > 解方程时两边同加 9 没问题，可这里只有一边。只加 9 就变成了另一条抛物线。

   Add AND subtract 9: $y = (x^2 - 6x + 9) - 9 + 2 = (x-3)^2 - 7$. Check: $f(3) = 9 - 18 + 2 = -7$ ✓.

   > 加 9 再减 9：$y = (x^2 - 6x + 9) - 9 + 2 = (x-3)^2 - 7$。检验：$f(3) = 9 - 18 + 2 = -7$ ✓。

4. **line · 第 1 —** Taking out -1 divides EVERY term in the bracket by -1, so $+6x$ becomes $-6x$. Only the x^2 term was changed.

   > 提出 -1 就是括号里「每一项」都除以 -1，所以 $+6x$ 要变成 $-6x$。这里只改了 x^2 项。

   $y = -(x^2 - 6x) + 1 = -(x^2 - 6x + 9 - 9) + 1 = -(x-3)^2 + 9 + 1 = -(x-3)^2 + 10$. Check: $p = -\frac{6}{2(-1)} = 3$ ✓.

   > $y = -(x^2 - 6x) + 1 = -(x^2 - 6x + 9 - 9) + 1 = -(x-3)^2 + 9 + 1 = -(x-3)^2 + 10$。检验：$p = -\frac{6}{2(-1)} = 3$ ✓。

5. **line · 第 1 —** The middle term of $(x+k)^2$ is $2kx$, so k is HALF of the x-coefficient. $(x+4)^2$ would give $8x$.

   > $(x+k)^2$ 的中间项是 $2kx$，所以 k 是 x 系数的「一半」。$(x+4)^2$ 会得到 $8x$。

   Half of 4 is 2: $y = (x+2)^2 - 4 - 1 = (x+2)^2 - 5$. Check: $f(-2) = 4 - 8 - 1 = -5$ ✓.

   > 4 的一半是 2：$y = (x+2)^2 - 4 - 1 = (x+2)^2 - 5$。检验：$f(-2) = 4 - 8 - 1 = -5$ ✓。