Max/min problems

最值应用题

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** What changes, and by how much each step (price step, sales lost, metres of fence)? Name the variable and say what it counts, e.g. "n = number of 1-dollar increases".

   > 什么在变？每一步变多少（涨价幅度、少卖多少、多少米篱笆）？设好变量并说清它数的是什么，比如「n = 涨 1 加元的次数」。

2. 🎯 **Asked · 求什么** Is it asking WHEN or WHICH value (the x-coordinate of the vertex), HOW MUCH (the y-coordinate), or both? In what units?

   > 问的是「何时 / 取何值」（顶点横坐标）、「是多少」（顶点纵坐标），还是两样都要？单位是什么？

3. ✏️ **Picture it · 画出来** Sketch the shape (a pen against a wall, three widths) or make a small table of n, price and number sold. The graph is a downward parabola; the answer is at the top.

   > 画出形状（靠墙的围栏、三条宽），或者列一个 n、单价、销量的小表。图象是开口向下的抛物线，答案在最高点。

4. 🧰 **Tool · 用什么工具** Write the quantity as a product of two linear factors, e.g. (price)(number sold) or (width)(length). Find where each factor is zero; the vertex is halfway between. If it is already expanded, use $-\dfrac{b}{2a}$.

   > 把要求的量写成两个一次因式的乘积，比如（单价）×（销量）或（宽）×（长）。求出每个因式为零的点，顶点就在两点正中间。如果已经展开了，就用 $-\dfrac{b}{2a}$。

5. ✅ **Check · 检验** Try one step either side of your answer: both must give less. Then answer the question asked in a sentence with units, and make sure it fits the story (price and sides positive).

   > 在答案两侧各试一步：两边都必须更小。然后用带单位的一句话回答题目真正问的，并看它是否符合情境（单价、边长为正）。

⭐ Must know

必须记住

- Optimize = find the vertex. "When / which value" → x-coordinate; "how much" → y-coordinate.
  > 求最值 = 求顶点。「何时 / 取何值」→ 横坐标；「最多 / 最少是多少」→ 纵坐标。
- Build the function first: area $= lw$, revenue = price × number sold.
  > 先建函数：面积 = 长 × 宽，收入 = 单价 × 销量。
- Product of two linear factors: best x = midpoint of their zeros.
  > 两个一次因式相乘：最优 x = 两个零点的中点。

$$x = -\frac{b}{2a}$$

$$R = (\text{price}) \times (\text{quantity})$$

🔍 Watch out for

当心

1. Answering with the wrong coordinate
   > 答错了坐标
2. Giving n as the price
   > 把 n 当成了价格
3. Fencing the side that has a wall
   > 靠墙的那边也算了篱笆
4. Dropping the 2 in the vertex formula
   > 顶点公式漏了分母里的 2
5. Losing 1 sale per step instead of the given drop
   > 每涨一次只减 1 个，忘了题给的减少量

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can expand $(5+n)(40-2n)$, compute $-\frac{b}{2a}$ with negatives, and find the zeros of a product.

> 我会展开 $(5+n)(40-2n)$，会算带负数的 $-\frac{b}{2a}$，也会求乘积式的零点。

(5′)

○ I can ask the five starting questions for a max/min word problem.

> 我能对最值应用题问出五个入手问题。

(5′)

○ I can follow, then finish, a revenue problem: name n, write (price)(number sold), take the midpoint of the zeros.

> 我能看懂并补全一道收入题：设 n，写成（单价）×（销量），取两个零点的中点。

(9′)

○ I can solve a revenue problem on my own with at most two hints and answer in a sentence with units.

> 我能在最多两个提示下独立解一道收入题，并用带单位的一句话作答。

(9′)

○ I can spot the five classic mistakes in someone else's work.

> 我能在别人的解答里找出五个经典错误。

(8′)

○ I can do the Level 1–2 practice questions.

> 我能做出 1–2 级练习题。

(10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Expand $(5+n)(40-2n)$.

> 展开 $(5+n)(40-2n)$。

A. $-2n^2+30n+200$ B. $-2n^2+50n+200$ C. $2n^2+30n+200$ D. $-2n^2+30n+45$

If you missed it · 如果做错了 — **Distributing (and the minus sign)**

> **去括号（尤其是负号）**

: Multiply EVERY term inside the bracket: $3(x-4)=3x-12$. A minus in front changes every sign inside: $-(x-5)=-x+5$, and $2-(x+3)=2-x-3=-x-1$.

> 括号里的每一项都要乘到：$3(x-4)=3x-12$。前面是负号时，括号里每一项都变号：$-(x-5)=-x+5$，$2-(x+3)=2-x-3=-x-1$。

1. $-2(3x-1)=-6x+2$ (the -1 becomes $+2$).
   > $-2(3x-1)=-6x+2$（-1 变成 $+2$）。
2. $x(x+2)-3(x+2)=x^2+2x-3x-6=x^2-x-6$.
   > $x(x+2)-3(x+2)=x^2+2x-3x-6=x^2-x-6$。

2. Find $-\dfrac{b}{2a}$ when $a=-4$ and $b=24$.

> 当 $a=-4$、$b=24$ 时，求 $-\dfrac{b}{2a}$。

A. 3 B. -3 C. 6 D. -6

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

> 减去负数等于加上它：$5-(-3)=5+3=8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2=16$，但 $-4^2=-16$（先算平方）。

1. $-7-(-2)$: change "minus negative" to "plus": $-7+2=-5$.
   > $-7-(-2)$：把「减负」改成「加」：$-7+2=-5$。
2. $(-3)(-5)=15$ (same signs), $(-3)(5)=-15$ (different signs).
   > $(-3)(-5)=15$（同号），$(-3)(5)=-15$（异号）。
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.
   > 把 $x=-3$ 代入 x^2：先加括号，$(-3)^2=9$。

3. For which values of x is $(x+6)(18-3x)=0$?

> x 取什么值时 $(x+6)(18-3x)=0$?

A. $x=-6$ or $x=6$ B. $x=6$ or $x=-18$ C. $x=-6$ or $x=18$ D. $x=6$ or $x=3$

If you missed it · 如果做错了 — **Factoring x^2+bx+c (product and sum)**

> **分解 x^2+bx+c（积与和）**

: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: x^2-x-12 → -4 and 3 → $(x-4)(x+3)$. Check by expanding.

> 先提公因式。再找两个数，相乘等于 c、相加等于 b：x^2-x-12 → -4 和 3 → $(x-4)(x+3)$。展开检验。

1. $x^2+7x+10$: $2\times5=10$, $2+5=7$ → $(x+2)(x+5)$.
   > $x^2+7x+10$：$2\times5=10$，$2+5=7$ → $(x+2)(x+5)$。
2. If $c<0$ the two numbers have different signs; if $c>0$ they share the sign of b.
   > 若 $c<0$，两个数异号；若 $c>0$，两个数与 b 同号。

4. A fence satisfies $2w+l=100$. Write l in terms of w.

> 篱笆满足 $2w+l=100$。用 w 表示 l。

A. $l=100-2w$ B. $l=50-w$ C. $l=100+2w$ D. $l=\dfrac{100-w}{2}$

If you missed it · 如果做错了 — **Rearranging a formula**

> **公式变形（求某个字母）**

: Treat the other letters as numbers and undo operations in reverse order. $A=P(1+r)$ for r: divide by P, then subtract 1: $r=\frac{A}{P}-1$.

> 把其他字母当成数字，按相反顺序撤销运算。由 $A=P(1+r)$ 求 r：先除以 P，再减 1：$r=\frac{A}{P}-1$。

1. $y=a(x-p)^2+q$ for a: subtract q, divide by $(x-p)^2$: $a=\frac{y-q}{(x-p)^2}$.
   > 由 $y=a(x-p)^2+q$ 求 a：减 q，再除以 $(x-p)^2$：$a=\frac{y-q}{(x-p)^2}$。
2. Substitute numbers first if the question gives them — fewer letters, fewer slips.
   > 题目给了数字就先代入——字母越少，越不容易错。

Answers · 答案: 1: A · 2: A · 3: A · 4: A

🔍 Common-errors sheet

常见错误单

1. Answering with the wrong coordinate

答错了坐标

A ball's height is $h(t) = -5t^2 + 20t + 1$ (metres, t in seconds). What is its maximum height?

小球高度 $h(t) = -5t^2 + 20t + 1$（米，t 为秒）。最大高度是多少?

1 $t = -\dfrac{20}{2(-5)} = 2.$

> $t = -\dfrac{20}{2(-5)} = 2$。

2 The maximum height is 2.

> 最大高度是 2。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

2. Giving n as the price

把 n 当成了价格

A club sells 150 shirts at 12 dollars each. Each 1-dollar increase loses 5 sales. What price maximizes revenue?

社团以每件 12 加元卖出 150 件 T 恤。每涨 1 加元，少卖 5 件。定价多少收入最高?

1 $R = (12 + n)(150 - 5n)$; zeros $n = -12$ and $n = 30$.

> $R = (12 + n)(150 - 5n)$；零点 $n = -12$ 和 $n = 30$。

2 Midpoint: $n = \dfrac{-12 + 30}{2} = 9.$

> 中点：$n = \dfrac{-12 + 30}{2} = 9$。

3 The best price is 9 dollars.

> 最佳定价是 9 加元。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

3. Fencing the side that has a wall

靠墙的那边也算了篱笆

100 m of fence makes a rectangular pen against a wall, fenced on three sides only. What is the largest area?

用 100 m 篱笆靠墙围一个长方形围栏，只围三边。最大面积是多少?

1 Perimeter: $2l + 2w = 100$, so $l = 50 - w$.

周长：$2l + 2w = 100$，所以 $l = 50 - w$。

2 $A = w(50 - w)$; zeros 0 and 50, so $w = 25$ and $A = 625$ m².

$A = w(50 - w)$；零点 0 和 50，所以 $w = 25$，$A = 625$ m²。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

4. Dropping the 2 in the vertex formula

顶点公式漏了分母里的 2

A ball's height is $h(t) = -4t^2 + 24t + 5$ (metres). Find the maximum height.

小球高度 $h(t) = -4t^2 + 24t + 5$（米）。求最大高度。

1 $t = -\dfrac{24}{-4} = 6.$

$t = -\dfrac{24}{-4} = 6$。

2 $h(6) = -144 + 144 + 5 = 5$ m.

$h(6) = -144 + 144 + 5 = 5$ m。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

5. Losing 1 sale per step instead of the given drop

每涨一次只减 1 个，忘了题给的减少量

A museum has 400 visitors a day at 14 dollars each. Each 1-dollar increase loses 20 visitors. What price maximizes revenue?

博物馆每张门票 14 加元时每天有 400 名游客。每涨 1 加元，少 20 名游客。票价定多少收入最高?

1 Let n = number of 1-dollar increases. Price $14 + n$, visitors $400 - n$.

设 n = 涨 1 加元的次数。票价 $14 + n$，游客 $400 - n$。

2 $R = (14 + n)(400 - n)$; zeros -14 and 400, so $n = 193$.

$R = (14 + n)(400 - n)$；零点 -14 和 400，所以 $n = 193$。

3 Price $14 + 193 = 207$ dollars.

票价 $14 + 193 = 207$ 加元。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么?

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** $t = 2$ tells you WHEN the top happens. The question asks HOW HIGH, which is the h-value at that time.

   > $t = 2$ 说的是「什么时候」到最高点。题目问的是「多高」，要的是那一刻的 h 值。

   $h(2) = -20 + 40 + 1 = 21$. The maximum height is 21 m, reached after 2 s.

   > $h(2) = -20 + 40 + 1 = 21$。最大高度 21 m，在第 2 秒到达。

2. **line · 第 3 —** n counts the 1-dollar INCREASES, not the price. A price of 9 dollars would be a price cut, which the model never described.

   > n 数的是涨价的「次数」，不是价格。9 加元等于降价了，模型里根本没有这回事。

   Price $= 12 + 9 = 21$ dollars, sales $150 - 45 = 105$, revenue $21 \times 105 = 2205$ dollars.

   > 价格 $= 12 + 9 = 21$ 加元，销量 $150 - 45 = 105$ 件，收入 $21 \times 105 = 2205$ 加元。

3. **line · 第 1 —** The wall is free; only two widths and ONE length use fence. Using the full perimeter spends fence on a side that needs none.

   > 墙是现成的，只有两条宽和「一条」长要用篱笆。用整个周长等于在不需要的边上也花了篱笆。

   $2w + l = 100$, $l = 100 - 2w$, $A = w(100 - 2w)$; zeros 0 and 50, $w = 25$, $l = 50$, $A = 1250$ m².

   > $2w + l = 100$，$l = 100 - 2w$，$A = w(100 - 2w)$；零点 0 和 50，$w = 25$，$l = 50$，$A = 1250$ m²。

4. **line · 第 1 —** The formula is $-\dfrac{b}{2a}$. Without the 2 you land at the time the ball comes back to its starting height, which is the clue: the "maximum" equals the starting height 5.

   > 公式是 $-\dfrac{b}{2a}$。漏了 2，就落在小球回到起始高度的时刻——破绽就在这里：「最大高度」竟然等于起始高度 5。

   $t = -\dfrac{24}{2(-4)} = 3$; $h(3) = -36 + 72 + 5 = 41$ m.

   > $t = -\dfrac{24}{2(-4)} = 3$；$h(3) = -36 + 72 + 5 = 41$ m。

5. **line · 第 1 —** Each increase loses 20 visitors, so after n increases $20n$ are gone. Writing $400 - n$ means only one visitor leaves per step, and the answer (207 dollars for a museum) fails the sense check.

   > 每涨一次少 20 人，涨 n 次就少 $20n$ 人。写成 $400 - n$ 等于每次只走一个人，结果（博物馆门票 207 加元）通不过合理性检查。

   Visitors $400 - 20n$. $R = (14 + n)(400 - 20n)$, zeros -14 and 20, $n = 3$: price 17 dollars, 340 visitors, $R = 5780$ dollars.

   > 游客 $400 - 20n$。$R = (14 + n)(400 - 20n)$，零点 -14 和 20，$n = 3$：票价 17 加元，游客 340 名，$R = 5780$ 加元。