Graphs & characteristics

图象与特征

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Write down a, b and c with their signs. The sign of a already tells you: up (minimum) or down (maximum).
   > 先写下带符号的 a、b、c。a 的符号已经告诉你：开口向上（有最小值）还是向下（有最大值）。
2. 🎯 **Asked · 求什么** Which of the six facts are wanted? Remember that a max or min VALUE is a y-value, and the range is about y.
   > 题目要六项特征中的哪几项？记住：最大值或最小值是一个 y 值，值域讲的也是 y。
3. ✏️ **Picture it · 画出来** A quick U or ∩ with a dashed vertical line for the axis. The vertex sits on that line, and the two x-intercepts are mirror images across it.
   > 随手画个 U 或 ∩，再画一条竖直虚线当对称轴。顶点在这条线上，两个 x 截距关于它对称。
4. 🧰 **Tool · 用什么工具** Axis first: $x = -\dfrac{b}{2a}$ (or the midpoint of the zeros). Put that x back into f to get the vertex height.
   > 先求对称轴：$x = -\dfrac{b}{2a}$（或两个零点的中点）。把这个 x 代回 f，得到顶点的高度。
5. ✅ **Check · 检验** The zeros must be equally far from the axis. The range must use the vertex y-value, with $\geq$ for up and $\leq$ for down.
   > 两个零点到对称轴的距离必须相等。值域要用顶点的纵坐标：开口向上用 $\geq$，向下用 $\leq$。

⭐ Must know

必须记住

- $a > 0$ opens up (minimum), $a < 0$ opens down (maximum); the vertex sits on the axis $x = -\frac{b}{2a}$.
  > $a > 0$ 开口向上（有最小值），$a < 0$ 开口向下（有最大值）；顶点在对称轴 $x = -\frac{b}{2a}$ 上。
- Range uses the vertex y-value: $y \geq q$ (up) or $y \leq q$ (down). Domain is always all real numbers.
  > 值域看顶点的纵坐标：开口向上 $y \geq q$，向下 $y \leq q$。定义域永远是全体实数。
- Axis = midpoint of the zeros, or of any two points at the same height.
  > 对称轴 = 两个零点的中点，或任意两个等高点的中点。

$$x = -\frac{b}{2a}$$

$$x_{\text{axis}} = \frac{r + s}{2}$$

🔍 Watch out for

当心

1. Losing the minus sign in the axis
   > 对称轴丢了负号
2. Writing the range with the x-value of the vertex
   > 用顶点横坐标写值域
3. Range in the wrong direction
   > 值域方向写反
4. Squaring a negative input without brackets
   > 代入负数平方时没加括号
5. Taking a given point as the top
   > 把题目给的点当成最高点

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can square a negative number with brackets, find the midpoint of two numbers, and factor $x^2 - 2x - 8$.

  > 我会给负数加括号再平方，会求两个数的中点，也会分解 $x^2 - 2x - 8$。

  (5′)

- ○ I can ask the five starting questions for a parabola in standard form.

  > 我能对一般式的抛物线问出五个入手问题。

  (5′)

- ○ I can follow, then finish, an axis-first solution that gives all six characteristics.

  > 我能看懂并补全「先求对称轴」的解法，得出全部六项特征。

  (8′)

- ○ I can find the vertex and range of a parabola on my own with at most two hints.

  > 我能在最多两个提示下独立求出抛物线的顶点和值域。

  (8′)

- ○ I can spot the five classic mistakes in someone else's work.

  > 我能在别人的解答里找出五个经典错误。

  (8′)

- ○ I can do the Level 1–2 practice questions.

  > 我能做出 1–2 级练习题。

  (10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Evaluate $-(-3)^2$.

> 求 $-(-3)^2$ 的值。

A. 9 B. -9 C. 6 D. -6

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

> 减去负数等于加上它：$5-(-3)=5+3=8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2=16$，但 $-4^2=-16$（先算平方）。

1. $-7-(-2)$: change "minus negative" to "plus": $-7+2=-5$.
   > $-7-(-2)$：把「减负」改成「加」：$-7+2=-5$。
2. $(-3)(-5)=15$ (same signs), $(-3)(5)=-15$ (different signs).
   > $(-3)(-5)=15$（同号），$(-3)(5)=-15$（异号）。
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.
   > 把 $x=-3$ 代入 x^2：先加括号，$(-3)^2=9$。

2. Evaluate $-\dfrac{-8}{2(2)}$.

> 求 $-\dfrac{-8}{2(2)}$ 的值。

A. 2 B. -2 C. 4 D. -4

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
   > $\frac{2}{3}-\frac{1}{4}$：公分母 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$。
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
   > $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$。
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$。

3. What number is exactly halfway between -5 and 1?

> 正好在 -5 和 1 中间的数是多少?

A. -3 B. -2 C. 2 D. 3

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

> 减去负数等于加上它：$5-(-3)=5+3=8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2=16$，但 $-4^2=-16$（先算平方）。

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   > $(-3)(-5)=15$（同号），$(-3)(5)=-15$（异号）。
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.
   > 把 $x=-3$ 代入 x^2：先加括号，$(-3)^2=9$。

4. Solve $x^2-2x-8=0$.

> 解 $x^2-2x-8=0$。

A. $x=-4$ or $x=2$ B. $x=4$ or $x=-2$ C. $x=8$ or $x=-1$ D. $x=2$ or $x=4$

If you missed it · 如果做错了 — **Factoring x^2+bx+c (product and sum)**

> **分解 x^2+bx+c（积与和）**

: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: x^2-x-12 → -4 and 3 → $(x-4)(x+3)$. Check by expanding.

> 先提公因式。再找两个数，相乘等于 c、相加等于 b：x^2-x-12 → -4 和 3 → $(x-4)(x+3)$。展开检验。

1. $x^2+7x+10$: $2\times5=10$, $2+5=7$ → $(x+2)(x+5)$.
   > $x^2+7x+10$：$2\times5=10$，$2+5=7$ → $(x+2)(x+5)$。
2. If $c<0$ the two numbers have different signs; if $c>0$ they share the sign of b.
   > 若 $c<0$，两个数异号；若 $c>0$，两个数与 b 同号。

Answers · 答案: 1: B · 2: A · 3: B · 4: B

🔍 Common-errors sheet

常见错误单

1. Losing the minus sign in the axis

对称轴丢了负号

Find the vertex of $y = x^2 + 8x + 7$.

求 $y = x^2 + 8x + 7$ 的顶点。

1 Axis: $x = \dfrac{b}{2a} = \dfrac{8}{2} = 4$

> 对称轴：$x = \dfrac{b}{2a} = \dfrac{8}{2} = 4$

2 $y = 16 + 32 + 7 = 55$

3 Vertex $(4, 55)$

> 顶点 $(4, 55)$

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Writing the range with the x-value of the vertex

用顶点横坐标写值域

Find the range of $y = x^2 - 10x + 21$.

求 $y = x^2 - 10x + 21$ 的值域。

1 Axis: $x = -\dfrac{-10}{2} = 5$

> 对称轴：$x = -\dfrac{-10}{2} = 5$

2 $y = 25 - 50 + 21 = -4$, vertex $(5, -4)$

> $y = 25 - 50 + 21 = -4$，顶点 $(5, -4)$

3 Opens up, so the range is $y \geq 5$.

> 开口向上，所以值域是 $y \geq 5$。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Range in the wrong direction

值域方向写反

Find the range of $f(x) = -3x^2 + 6x + 1$.

求 $f(x) = -3x^2 + 6x + 1$ 的值域。

1 Axis: $x = -\dfrac{6}{2(-3)} = 1$

对称轴：$x = -\dfrac{6}{2(-3)} = 1$

2 $f(1) = -3 + 6 + 1 = 4$

3 Range: $y \geq 4$

值域：$y \geq 4$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Squaring a negative input without brackets

代入负数平方时没加括号

For $f(x) = -x^2 + 4x + 3$, find $f(-2)$.

对于 $f(x) = -x^2 + 4x + 3$，求 $f(-2)$。

1 $f(-2) = -(-2)^2 + 4(-2) + 3$

2 $= 4 - 8 + 3$

3 $= -1$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Taking a given point as the top

把题目给的点当成最高点

A parabola opens down, has x-intercepts 0 and 8, and passes through $(2, 6)$. Find its maximum value.

一条开口向下的抛物线，x 截距为 0 和 8，且过点 $(2, 6)$。求它的最大值。

1 $y = ax(x - 8)$; from $(2, 6)$: $6 = a(2)(-6)$, so $a = -\frac{1}{2}$

$y = ax(x - 8)$；由 $(2, 6)$：$6 = a(2)(-6)$，得 $a = -\frac{1}{2}$

2 The highest point we know is $(2, 6)$, so the maximum is 6.

已知的最高点是 $(2, 6)$，所以最大值是 6。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 1 —** The formula is $x = -\dfrac{b}{2a}$. With b positive the axis is on the negative side.

   > 公式是 $x = -\dfrac{b}{2a}$。b 为正时，对称轴在负的一侧。

   $x = -\dfrac{8}{2} = -4$, then $y = 16 - 32 + 7 = -9$.

   > $x = -\dfrac{8}{2} = -4$，再算 $y = 16 - 32 + 7 = -9$。

2. **line · 第 3 —** The range lists y-values, so it must use the height of the vertex, not where it sits left or right.

   > 值域列的是 y 值，要用顶点的高度，而不是它在左右哪个位置。

   The lowest y is -4, so the range is $y \geq -4$.

   > 最小的 y 是 -4，所以值域是 $y \geq -4$。

3. **line · 第 3 —** Habit: most parabolas you have drawn open up. Here $a = -3 < 0$, so the vertex is the TOP and every other y is below it.

   > 习惯作怪：画过的抛物线大多开口向上。这里 $a = -3 < 0$，顶点是「最高点」，其他 y 都比它低。

   Opens down, so the range is $y \leq 4$.

   > 开口向下，值域是 $y \leq 4$。

4. **line · 第 2 —** The two minus signs looked like they cancel. But the square acts first: $(-2)^2 = 4$, and then the minus in front makes it -4.

   > 两个负号看起来像能抵消。但平方先算：$(-2)^2 = 4$，前面的负号再把它变成 -4。

   $-4 - 8 + 3 = -9$.

   > $-4 - 8 + 3 = -9$。

5. **line · 第 2 —** The top is always on the axis. The point $(2, 6)$ only helps find a; it is not on the axis $x = 4$.

   > 最高点一定在对称轴上。点 $(2, 6)$ 只用来求 a，它不在对称轴 $x = 4$ 上。

   Axis $x = \dfrac{0 + 8}{2} = 4$; $y = -\frac{1}{2}(4)(-4) = 8$.

   > 对称轴 $x = \dfrac{0 + 8}{2} = 4$；$y = -\frac{1}{2}(4)(-4) = 8$。