Rational equations & applications

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** Factor every denominator and write the NPVs on line 1. In a word problem, define the variable in words and add the context limits (time and speed > 0, people a whole number).
2. 🎯 **Asked** Every value of x that makes the ORIGINAL equation true and is allowed. There may be one, two, or none.
3. ✏️ **Picture it** For word problems, a small table: one row per person, pipe or trip; columns for rate (or speed) and time. The equation comes from one column.
4. 🧰 **Tool** Multiply EVERY term on both sides — whole numbers too — by the LCD. Solve the linear or quadratic equation left over.
5. ✅ **Check** Cross out any answer that is an NPV or breaks the context. Put the survivors back into the original equation.

⭐ Must know

- NPVs first; then multiply EVERY term (whole numbers too) by the LCD.
- An answer equal to an NPV is extraneous — reject it; all rejected → no solution.
- Work: rates add, $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$. Travel: time $= \frac{\text{distance}}{\text{speed}}$.

$$\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$$

$$t = \frac{d}{v}$$

🔍 Watch out for

1. Keeping the root that is not allowed
2. Not multiplying the whole-number term
3. Losing the minus in front of a fraction
4. Adding times instead of rates
5. Subtracting the times the wrong way round

🌱 Small wins — tick as you go

- ○ I can find an LCD like $(x-2)(x+2)$, solve a linear equation with brackets, and solve a quadratic by factoring. (6′)
- ○ I can write the NPVs on line 1 and decide the LCD before I multiply anything. (6′)
- ○ I can follow, then finish, the NPV–multiply–solve–reject method on an equation with one extraneous root. (10′)
- ○ I can solve a rational equation with a whole-number term on my own, with at most two hints. (8′)
- ○ I can spot the five classic rational-equation mistakes, including in work and travel problems. (8′)
- ○ I can do the Level 1–3 practice equations and say why each rejected root is rejected. (12′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. What is the lowest common denominator of $\frac{1}{x-2}$ and $\frac{3}{x^2-4}$?
A. $(x-2)(x^2-4)$ B. $(x-2)(x+2)$ C. x^2-2 D. $(x-2)^2(x+2)$

If you missed it — **Lowest common denominator**: Factor each denominator; the LCD uses every factor the HIGHEST number of times it appears. For 6 and 4: $2 \cdot 3$ and $2^2 \to 2^2 \cdot 3 = 12$. For x and x^2: x^2. For $(x+1)$ and $(x-2)$: $(x+1)(x-2)$.

1. $\frac{1}{2x}$ and $\frac{1}{3x^2}$: LCD $= 6x^2$.
2. Multiply top AND bottom by what is missing: $\frac{1}{2x} = \frac{3x}{6x^2}$.

2. Solve $3(x-2) = x+4$.
A. $x=1$ B. $x=3$ C. $x=5$ D. $x=-5$

If you missed it — **Solving a linear equation**: Do the same thing to both sides. Clear brackets, collect x on one side and numbers on the other, then divide. With fractions, multiply every term by the common denominator first.

1. $3(x-2) = x+8 \to 3x-6 = x+8 \to 2x = 14 \to x = 7$.
2. $\frac{x}{2} + \frac{x}{3} = 5$: multiply every term by 6 $\to 3x + 2x = 30 \to x = 6$.
3. Check by substituting: $3(7-2) = 15$ and $7+8 = 15$ ✓.

3. Solve $x^2 - x - 12 = 0$.
A. $x=4$ or $x=-3$ B. $x=-4$ or $x=3$ C. $x=6$ or $x=-2$ D. $x=12$ or $x=-1$

If you missed it — **Factoring x^2+bx+c (product and sum)**: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: $x^2-x-12 \to -4$ and $3 \to (x-4)(x+3)$. Check by expanding.

1. $x^2+7x+10$: $2 \times 5 = 10$, $2+5 = 7 \to (x+2)(x+5)$.
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.

4. A tap fills a tank in 5 hours. What fraction of the tank does it fill in 1 hour?
A. 5 B. $\frac{1}{2}$ C. $\frac{1}{10}$ D. $\frac{1}{5}$

If you missed it — **Percent, decimals and rates**: Percent to decimal: divide by 100 ($4.5\% = 0.045$). A rate per year split into n periods is $\frac{r}{n}$ per period: 6% compounded monthly is 0.005 per month. Increase by r: multiply by $(1+r)$.

1. $\$200$ increased by 3%: $200 \times 1.03 = 206$.
2. 3.6% compounded quarterly: $\frac{0.036}{4} = 0.009$ per quarter.

Answers: 1: B · 2: C · 3: A · 4: D

🔍 Common-errors sheet

1. Keeping the root that is not allowed

Solve $\dfrac{x^2}{x-2} = \dfrac{4}{x-2}$.

1 Multiply by $x-2$: $x^2 = 4$

2 $x = 2$ or $x = -2$, both answers.

✍ Which line goes wrong first, and why?

2. Not multiplying the whole-number term

Solve $x + \dfrac{4}{x} = 5$.

1 Multiply by x: $x^2 + 4 = 5$

2 $x^2 = 1$, so $x = \pm 1$

✍ Which line goes wrong first, and why?

3. Losing the minus in front of a fraction

Solve $\dfrac{3}{x} - \dfrac{x-6}{x} = 2$.

1 Multiply by x: $3 - x - 6 = 2x$

2 $-3 = 3x$, so $x = -1$

✍ Which line goes wrong first, and why?

4. Adding times instead of rates

One printer finishes a job in 4 h, another in 12 h. How long working together?

1 Together: $4 + 12 = 16$ h

2 Answer: 16 h

✍ Which line goes wrong first, and why?

5. Subtracting the times the wrong way round

A runner covers 12 km. Running 2 km/h faster would take 1 h less. Find her speed.

1 $\dfrac{12}{v+2} - \dfrac{12}{v} = 1$

2 $12v - 12(v+2) = v(v+2)$, so $v^2 + 2v + 24 = 0$

3 No real solution.

✍ Which line goes wrong first, and why?

Key (fold this part back)

1. **line 2** — The NPVs were never written down, so nothing reminded the solver that $x = 2$ makes the denominator 0. NPV $x \neq 2$ on line 1. Then $x^2 = 4$ gives ± 2; reject 2.
2. **line 1** — Only the terms that looked like fractions got multiplied. But the LCD must multiply EVERY term, and the 5 is a term too. $x^2 + 4 = 5x$, so $x^2 - 5x + 4 = 0$, $(x-1)(x-4) = 0$: $x = 1$ or $x = 4$ (neither is the NPV 0).
3. **line 1** — When the fraction disappeared, its numerator lost its brackets. The minus must hit the whole numerator: $-(x-6) = -x + 6$. $3 - (x-6) = 2x$, so $9 - x = 2x$, $x = 3$. Check: $1 - \frac{-3}{3} = 2$ ✓.
4. **line 1** — Two machines together must be FASTER than either alone, so the answer has to be under 4 h. What adds is the part of the job done each hour. $\frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{1}{3}$ job per hour, so together $T = 3$ h.
5. **line 1** — The faster trip takes LESS time, so $\frac{12}{v+2}$ is the smaller time. Slower time minus faster time is what equals 1. $\frac{12}{v} - \frac{12}{v+2} = 1$ gives $24 = v^2 + 2v$, $(v+6)(v-4) = 0$, $v = 4$ (reject -6). Check: 3 h vs 2 h ✓.