Rational equations & applications

分式方程与应用题

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Factor every denominator and write the NPVs on line 1. In a word problem, define the variable in words and add the context limits (time and speed > 0, people a whole number).
   > 把每个分母分解，第一行写出不允许值。应用题先用文字定义变量，再加上实际限制（时间、速度 > 0，人数是整数）。
2. 🎯 **Asked · 求什么** Every value of x that makes the ORIGINAL equation true and is allowed. There may be one, two, or none.
   > 所有让「原」方程成立而且被允许的 x——可能一个、两个，也可能没有。
3. ✏️ **Picture it · 画出来** For word problems, a small table: one row per person, pipe or trip; columns for rate (or speed) and time. The equation comes from one column.
   > 应用题画一个小表：每个人、每根管或每一段路一行；列是效率（或速度）和时间。方程来自其中一列。
4. 🧰 **Tool · 用什么工具** Multiply EVERY term on both sides — whole numbers too — by the LCD. Solve the linear or quadratic equation left over.
   > 把两边「每一项」——包括整数项——都乘以最简公分母，再解剩下的一次或二次方程。
5. ✅ **Check · 检验** Cross out any answer that is an NPV or breaks the context. Put the survivors back into the original equation.
   > 划掉等于不允许值或不符合实际的答案，把剩下的代回原方程检验。

⭐ Must know

必须记住

- NPVs first; then multiply EVERY term (whole numbers too) by the LCD.
  > 先写不允许值；再把每一项（包括整数项）都乘以公分母。
- An answer equal to an NPV is extraneous — reject it; all rejected → no solution.
  > 等于不允许值的解是增根，舍去；全部舍去 → 无解。
- Work: rates add, $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$. Travel: time $= \frac{\text{distance}}{\text{speed}}$.
  > 工作问题：效率相加，$\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$。行程问题：时间 $= \frac{\text{路程}}{\text{速度}}$。

$$\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{T}$$

$$t = \frac{d}{v}$$

🔍 Watch out for

当心

1. Keeping the root that is not allowed
   > 留下了不允许的根
2. Not multiplying the whole-number term
   > 整数项没乘公分母
3. Losing the minus in front of a fraction
   > 分式前面的减号丢了
4. Adding times instead of rates
   > 把时间相加，而不是把效率相加
5. Subtracting the times the wrong way round
   > 时间相减的顺序弄反了

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can find an LCD like $(x-2)(x+2)$, solve a linear equation with brackets, and solve a quadratic by factoring.

> 我会找 $(x-2)(x+2)$ 这样的最简公分母，会解带括号的一次方程，会用因式分解解二次方程。

(6′)

○ I can write the NPVs on line 1 and decide the LCD before I multiply anything.

> 我能在第一行写出不允许值，并在乘任何东西之前定好最简公分母。

(6′)

○ I can follow, then finish, the NPV–multiply–solve–reject method on an equation with one extraneous root.

> 我能看懂并补全「不允许值—乘公分母—求解—舍根」的方法，处理带一个增根的方程。

(10′)

○ I can solve a rational equation with a whole-number term on my own, with at most two hints.

> 我能在最多两个提示下，独立解一道含整数项的分式方程。

(8′)

○ I can spot the five classic rational-equation mistakes, including in work and travel problems.

> 我能找出分式方程的五个经典错误，包括工程和行程应用题里的。

(8′)

○ I can do the Level 1–3 practice equations and say why each rejected root is rejected.

> 我能做出 1–3 级的方程练习，并说出每个被舍去的根为什么要舍。

(12′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. What is the lowest common denominator of $\frac{1}{x-2}$ and $\frac{3}{x^2-4}$?

> $\frac{1}{x-2}$ 和 $\frac{3}{x^2-4}$ 的最简公分母是什么?

A. $(x-2)(x^2-4)$ B. $(x-2)(x+2)$ C. x^2-2 D. $(x-2)^2(x+2)$

If you missed it · 如果做错了 — **Lowest common denominator**

> **最简公分母**

: Factor each denominator; the LCD uses every factor the HIGHEST number of times it appears. For 6 and 4: $2\cdot3$ and 2^2 → $2^2\cdot3=12$. For x and x^2: x^2. For $(x+1)$ and $(x-2)$: $(x+1)(x-2)$.

> 把每个分母分解；最简公分母取每个因式出现的最高次数。6 和 4：$2\cdot3$ 与 2^2 → $2^2\cdot3=12$。x 和 x^2：x^2。$(x+1)$ 和 $(x-2)$：$(x+1)(x-2)$。

1. $\frac{1}{2x}$ and $\frac{1}{3x^2}$: LCD $=6x^2$.
   > $\frac{1}{2x}$ 与 $\frac{1}{3x^2}$：最简公分母 $=6x^2$。
2. Multiply top AND bottom by what is missing: $\frac{1}{2x}=\frac{3x}{6x^2}$.
   > 分子分母同乘缺的部分：$\frac{1}{2x}=\frac{3x}{6x^2}$。

2. Solve $3(x-2)=x+4$.

> 解 $3(x-2)=x+4$。

A. $x=1$ B. $x=3$ C. $x=5$ D. $x=-5$

If you missed it · 如果做错了 — **Solving a linear equation**

> **解一元一次方程**

: Do the same thing to both sides. Clear brackets, collect x on one side and numbers on the other, then divide. With fractions, multiply every term by the common denominator first.

> 两边做同样的事。先去括号，把含 x 的项移到一边、数字移到另一边，最后除以系数。有分数时先把每一项都乘以公分母。

1. $3(x-2)=x+8$ → $3x-6=x+8$ → $2x=14$ → $x=7$.
   > $3(x-2)=x+8$ → $3x-6=x+8$ → $2x=14$ → $x=7$。
2. $\frac{x}{2}+\frac{x}{3}=5$: multiply every term by 6 → $3x+2x=30$ → $x=6$.
   > $\frac{x}{2}+\frac{x}{3}=5$：每项乘 6 → $3x+2x=30$ → $x=6$。
3. Check by substituting: $3(7-2)=15$ and $7+8=15$ ✓.
   > 代回检验：$3(7-2)=15$，$7+8=15$ ✓。

3. Solve $x^2 - x - 12 = 0$.

> 解 $x^2 - x - 12 = 0$。

A. $x = 4$ or $x = -3$ B. $x = -4$ or $x = 3$ C. $x = 6$ or $x = -2$ D. $x = 12$ or $x = -1$

If you missed it · 如果做错了 — **Factoring $x^2 + bx + c$ (product and sum)**

> **分解 $x^2 + bx + c$（积与和）**

: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: $x^2 - x - 12$ → -4 and 3 → $(x - 4)(x + 3)$. Check by expanding.

> 先提公因式。再找两个数，相乘等于 c、相加等于 b：$x^2 - x - 12$ → -4 和 3 → $(x - 4)(x + 3)$。展开检验。

1. $x^2 + 7x + 10$: $2 \times 5 = 10$, $2 + 5 = 7$ → $(x + 2)(x + 5)$.
   > $x^2 + 7x + 10$：$2 \times 5 = 10$，$2 + 5 = 7$ → $(x + 2)(x + 5)$。
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.
   > 若 $c < 0$，两个数异号；若 $c > 0$，两个数与 b 同号。

4. A tap fills a tank in 5 hours. What fraction of the tank does it fill in 1 hour?

> 一个水龙头 5 小时灌满一个水箱。它 1 小时灌满水箱的几分之几?

A. 5 B. $\frac{1}{2}$ C. $\frac{1}{10}$ D. $\frac{1}{5}$

If you missed it · 如果做错了 — **Percent, decimals and rates**

> **百分数、小数与比率**

: Percent to decimal: divide by 100 ($4.5\% = 0.045$). A rate per year split into n periods is $\frac{r}{n}$ per period: 6% compounded monthly is 0.005 per month. Increase by r: multiply by $(1 + r)$.

> 百分数化小数：除以 100（$4.5\% = 0.045$）。年利率分成 n 期，每期是 $\frac{r}{n}$：年利率 6% 按月复利，每月 0.005。增长 r：乘以 $(1 + r)$。

1. $\$200$ increased by 3%: $200 \times 1.03 = 206$.
   > $\$200$ 增长 3%：$200 \times 1.03 = 206$。
2. 3.6% compounded quarterly: $\frac{0.036}{4} = 0.009$ per quarter.
   > 年利率 3.6% 按季复利：每季 $\frac{0.036}{4} = 0.009$。

Answers · 答案: 1: B · 2: C · 3: A · 4: D

🔍 Common-errors sheet

常见错误单

1. Keeping the root that is not allowed

留下了不允许的根

Solve $\dfrac{x^2}{x-2} = \dfrac{4}{x-2}$.

解方程 $\dfrac{x^2}{x-2} = \dfrac{4}{x-2}$。

1 Multiply by $x-2$: $x^2 = 4$

两边乘 $x-2$：$x^2 = 4$

2 $x = 2$ or $x = -2$, both answers.

$x = 2$ 或 $x = -2$，两个都是解。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Not multiplying the whole-number term

整数项没乘公分母

Solve $x + \dfrac{4}{x} = 5$.

解方程 $x + \dfrac{4}{x} = 5$。

1 Multiply by x: $x^2 + 4 = 5$

两边乘 x：$x^2 + 4 = 5$

2 $x^2 = 1$, so $x = \pm 1$

$x^2 = 1$，所以 $x = \pm 1$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Losing the minus in front of a fraction

分式前面的减号丢了

Solve $\dfrac{3}{x} - \dfrac{x-6}{x} = 2$.

解方程 $\dfrac{3}{x} - \dfrac{x-6}{x} = 2$。

1 Multiply by x: $3 - x - 6 = 2x$

两边乘 x：$3 - x - 6 = 2x$

2 $-3 = 3x$, so $x = -1$

$-3 = 3x$，所以 $x = -1$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Adding times instead of rates

> **把时间相加，而不是把效率相加**

One printer finishes a job in 4 h, another in 12 h. How long working together?

> 一台打印机 4 小时完成一项任务，另一台要 12 小时。两台一起要多久？

1 Together: $4 + 12 = 16$ h

> 一起：$4 + 12 = 16$ 小时

2 Answer: 16 h

> 答：16 小时

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Subtracting the times the wrong way round

> **时间相减的顺序弄反了**

A runner covers 12 km. Running 2 km/h faster would take 1 h less. Find her speed.

> 一名跑步者跑 12 km。如果每小时快 2 km，就能少用 1 小时。求她的速度。

1 $\dfrac{12}{v+2} - \dfrac{12}{v} = 1$

2 $12v - 12(v+2) = v(v+2)$, so $v^2 + 2v + 24 = 0$

> $12v - 12(v+2) = v(v+2)$，即 $v^2 + 2v + 24 = 0$

3 No real solution.

> 无实数解。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** The NPVs were never written down, so nothing reminded the solver that $x = 2$ makes the denominator 0.

   > 一开始没写不允许值，所以没人提醒 $x = 2$ 会让分母为 0。

   NPV $x \neq 2$ on line 1. Then $x^2 = 4$ gives ± 2; reject 2.

   > 第一行写 $x \neq 2$。$x^2 = 4$ 得 ± 2，舍去 2。

2. **line · 第 1 —** Only the terms that looked like fractions got multiplied. But the LCD must multiply EVERY term, and the 5 is a term too.

   > 只乘了看起来像分数的项。可是公分母要乘「每一项」，5 也是一项。

   $x^2 + 4 = 5x$, so $x^2 - 5x + 4 = 0$, $(x-1)(x-4) = 0$: $x = 1$ or $x = 4$ (neither is the NPV 0).

   > $x^2 + 4 = 5x$，即 $x^2 - 5x + 4 = 0$，$(x-1)(x-4) = 0$：$x = 1$ 或 $x = 4$（都不是不允许值 0）。

3. **line · 第 1 —** When the fraction disappeared, its numerator lost its brackets. The minus must hit the whole numerator: $-(x - 6) = -x + 6$.

   > 分母一消失，分子的括号也跟着丢了。减号要作用于整个分子：$-(x-6) = -x + 6$。

   $3 - (x - 6) = 2x$, so $9 - x = 2x$, $x = 3$. Check: $1 - \frac{-3}{3} = 2$ ✓.

   > $3 - (x-6) = 2x$，即 $9 - x = 2x$，$x = 3$。检验：$1 - \frac{-3}{3} = 2$ ✓。

4. **line · 第 1 —** Two machines together must be FASTER than either alone, so the answer has to be under 4 h. What adds is the part of the job done each hour.

   > 两台一起一定比任何一台单独快，所以答案必须少于 4 小时。能相加的是「每小时完成的部分」。

   $\frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{1}{3}$ job per hour, so together $T = 3$ h.

   > $\frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{1}{3}$（每小时完成的部分），所以一起 $T = 3$ 小时。

5. **line · 第 1 —** The faster trip takes LESS time, so $\frac{12}{v+2}$ is the smaller time. Slower time minus faster time is what equals 1.

   > 跑得快用时「少」，所以 $\frac{12}{v+2}$ 是较小的时间。应该是慢的时间减快的时间等于 1。

   $\frac{12}{v} - \frac{12}{v+2} = 1$ gives $24 = v^2 + 2v$, $(v+6)(v-4) = 0$, $v = 4$ (reject -6). Check: 3 h vs 2 h ✓.

   > $\frac{12}{v} - \frac{12}{v+2} = 1$ 得 $24 = v^2 + 2v$，$(v+6)(v-4) = 0$，$v = 4$（舍去 -6）。检验：3 小时与 2 小时 ✓。