Multiplying & dividing

有理式乘除

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Is it $\times$ or $\div$? Factor every numerator and every denominator before anything else.
   > 是乘还是除？先把每个分子、每个分母都分解，再做别的。
2. 🎯 **Asked · 求什么** One fraction in lowest terms, followed by EVERY non-permissible value.
   > 一个最简分式，后面写上「全部」不允许值。
3. ✏️ **Picture it · 画出来** Draw three boxes for NPVs: bottom of fraction 1, bottom of fraction 2, and (for division only) top of the divisor. Fill all three before you flip.
   > 画三个格子装不允许值：第 1 个分式的分母、第 2 个分式的分母，以及（只在除法时）除式的分子。颠倒之前把三个格子都填好。
4. 🧰 **Tool · 用什么工具** Division: keep the first fraction, change $\div$ to $\times$, flip the SECOND fraction. Then cancel any top factor with any bottom factor and multiply what is left.
   > 除法：第一个分式不变，$\div$ 变 $\times$，把「第二个」分式颠倒。然后任一分子因式都可以和任一分母因式约分，再把剩下的乘起来。
5. ✅ **Check · 检验** Test an allowed value such as $x = 0$ or $x = 1$ in the original and in your answer. Count your NPVs against the three boxes.
   > 用一个允许的值（如 $x = 0$ 或 $x = 1$）分别代入原式和答案。再对照三个格子数一数不允许值。

⭐ Must know

必须记住

- Multiply: factor everything, cancel any top with any bottom, then multiply what is left.
  > 乘法：全部分解，任一分子可与任一分母约分，再把剩下的相乘。
- Divide: multiply by the reciprocal of the SECOND fraction.
  > 除法：乘以第二个分式的倒数。
- Division NPVs: both denominators AND the numerator of the divisor.
  > 除法的不允许值：两个分母，再加上除式的分子。

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

🔍 Watch out for

当心

1. Missing the NPV from the divisor's numerator
   > 漏了除式分子给出的不允许值
2. Cancelling across a plus sign
   > 隔着加号约分
3. Losing the divisor's old denominator after flipping
   > 颠倒后丢了除式原来的分母
4. Cancelling a power completely
   > 把幂整个约光了
5. Flipping the wrong fraction
   > 颠倒错了分式

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can divide number fractions, factor a trinomial and simplify $\frac{x^5}{x^2}$.

> 我会做数字分数的除法、分解三项式，也会化简 $\frac{x^5}{x^2}$。

(5′)

○ I can ask the five starting questions for a product or quotient of rational expressions.

> 我能对有理式的乘法或除法问出五个入手问题。

(5′)

○ I can follow, then finish, a "factor, list NPVs from three places, flip, cancel" division.

> 我能看懂并补全「分解—从三处写不允许值—颠倒—约分」的除法。

(8′)

○ I can do a division on my own with at most two hints and get every NPV.

> 我能在最多两个提示下独立做一道除法，并写全不允许值。

(9′)

○ I can spot the five classic mistakes in someone else's work.

> 我能在别人的解答里找出五个经典错误。

(8′)

○ I can do the Level 1–2 practice questions.

> 我能做出 1–2 级练习题。

(10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Compute $\frac{2}{3} \div \frac{4}{9}$.

> 计算 $\frac{2}{3} \div \frac{4}{9}$。

A. $\frac{8}{27}$ B. $\frac{3}{2}$ C. $\frac{2}{3}$ D. $\frac{6}{12}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**
> : Add or subtract only with a common denominator: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).
> 加减必须先通分：$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3} - \frac{1}{4}$: common denominator 12 → $\frac{8}{12} - \frac{3}{12} = \frac{5}{12}$.
   > $\frac{2}{3} - \frac{1}{4}$：公分母 12 → $\frac{8}{12} - \frac{3}{12} = \frac{5}{12}$。
2. $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$.
   > $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$。
3. Never add denominators: $\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$。

2. Factor $x^2 + 2x - 15$.

> 分解因式 $x^2 + 2x - 15$。

A. $(x-5)(x+3)$ B. $(x+5)(x+3)$ C. $(x+5)(x-3)$ D. $(x+15)(x-1)$

If you missed it · 如果做错了 — **Factoring $x^2 + bx + c$ (product and sum)**

> **分解 $x^2 + bx + c$（积与和）**
> : Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: $x^2 - x - 12$ → -4 and 3 → $(x-4)(x+3)$. Check by expanding.
> 先提公因式。再找两个数，相乘等于 c、相加等于 b：$x^2 - x - 12$ → -4 和 3 → $(x-4)(x+3)$。展开检验。

1. $x^2 + 7x + 10$: $2 \times 5 = 10$, $2 + 5 = 7$ → $(x+2)(x+5)$.
   > $x^2 + 7x + 10$：$2 \times 5 = 10$，$2 + 5 = 7$ → $(x+2)(x+5)$。
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.
   > 若 $c < 0$，两个数异号；若 $c > 0$，两个数与 b 同号。

3. Simplify $\dfrac{x^5}{x^2}$.

> 化简 $\dfrac{x^5}{x^2}$。

A. x^3 B. $x^{2.5}$ C. x^7 D. x^{10}

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

> 同底数：相乘→指数相加（$x^3x^4 = x^7$）；相除→指数相减（$x^7 \div x^2 = x^5$）；幂的乘方→指数相乘（$(x^3)^2 = x^6$）。积的乘方每个因数都要乘方：$(2x)^3 = 8x^3$。

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
   > $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$。
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).
   > $\frac{6x^5}{2x^2} = 3x^3$（系数相除，指数相减）。

4. Compute $\dfrac{3}{4} \cdot \dfrac{8}{9}$.

> 计算 $\dfrac{3}{4} \cdot \dfrac{8}{9}$。

A. $\dfrac{11}{13}$ B. $\dfrac{27}{32}$ C. $\dfrac{2}{3}$ D. $\dfrac{3}{2}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3} - \frac{1}{4}$: common denominator 12 → $\frac{8}{12} - \frac{3}{12} = \frac{5}{12}$.
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   > $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$。
3. Never add denominators: $\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$。

Answers · 答案: 1: B · 2: C · 3: A · 4: C

🔍 Common-errors sheet

常见错误单

1. Missing the NPV from the divisor's numerator

> **漏了除式分子给出的不允许值**

Simplify $\dfrac{x-5}{x+2} \div \dfrac{x-5}{x-1}$ and state all NPVs.

> 化简 $\dfrac{x-5}{x+2} \div \dfrac{x-5}{x-1}$，并写出全部不允许值。

1 NPVs: the bottoms give $x \neq -2$ and $x \neq 1$. That is all of them.

> 不允许值：两个分母给出 $x \neq -2$，$x \neq 1$。就这些。

2 Flip and cancel: $\dfrac{x-5}{x+2} \cdot \dfrac{x-1}{x-5} = \dfrac{x-1}{x+2}$.

> 颠倒并约分：$\dfrac{x-5}{x+2} \cdot \dfrac{x-1}{x-5} = \dfrac{x-1}{x+2}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Cancelling across a plus sign

> **隔着加号约分**

Simplify $\dfrac{x+5}{x} \cdot \dfrac{2x}{x^2-25}$.

> 化简 $\dfrac{x+5}{x} \cdot \dfrac{2x}{x^2-25}$。

1 Cancel the x in $\dfrac{x+5}{x}$: it becomes 5.

> 把 $\dfrac{x+5}{x}$ 里的 x 约掉：变成 5。

2 $5 \cdot \dfrac{2x}{x^2-25} = \dfrac{10x}{x^2-25}$.

> $5 \cdot \dfrac{2x}{x^2-25} = \dfrac{10x}{x^2-25}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Losing the divisor's old denominator after flipping

颠倒后丢了除式原来的分母

Simplify $\dfrac{2x}{x-3} \div \dfrac{x+1}{x+4}$ and state all NPVs.

化简 $\dfrac{2x}{x-3} \div \dfrac{x+1}{x+4}$，并写出全部不允许值。

1 Flip: $\dfrac{2x}{x-3} \cdot \dfrac{x+4}{x+1} = \dfrac{2x(x+4)}{(x-3)(x+1)}$.

> 颠倒：$\dfrac{2x}{x-3} \cdot \dfrac{x+4}{x+1} = \dfrac{2x(x+4)}{(x-3)(x+1)}$。

2 NPVs from this bottom: $x \neq 3,\ -1$.

> 从这个分母找不允许值：$x \neq 3,\ -1$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Cancelling a power completely

把幂整个约光了

Simplify $\dfrac{x^2-9}{4x} \cdot \dfrac{8x^3}{x+3}$.

化简 $\dfrac{x^2-9}{4x} \cdot \dfrac{8x^3}{x+3}$。

1 Factor: $\dfrac{(x-3)(x+3)}{4x} \cdot \dfrac{8x^3}{x+3}$.

> 分解：$\dfrac{(x-3)(x+3)}{4x} \cdot \dfrac{8x^3}{x+3}$。

2 Cancel $(x+3)$, 4 into 8, and x into x^3: $2x(x-3)$.

> 约去 $(x+3)$，4 约进 8，x 约进 x^3：$2x(x-3)$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Flipping the wrong fraction

颠倒错了分式

Simplify $\dfrac{x}{x+3} \div \dfrac{x^2}{x^2-9}$ and state all NPVs.

化简 $\dfrac{x}{x+3} \div \dfrac{x^2}{x^2-9}$，并写出全部不允许值。

1 Flip the first one: $\dfrac{x+3}{x} \cdot \dfrac{x^2}{x^2-9}$.

把第一个颠倒：$\dfrac{x+3}{x} \cdot \dfrac{x^2}{x^2-9}$。

2 Cancel: $\dfrac{x}{x-3}$.

约分：$\dfrac{x}{x-3}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 1 —** After flipping, the divisor's top $x-5$ lands on the bottom. At $x=5$ you would be dividing by $\frac{0}{4}=0$, which is not allowed.

   颠倒以后，除式的分子 $x-5$ 跑到了分母。$x=5$ 时等于除以 $\frac{0}{4}=0$，不允许。

   Three boxes: $x \neq -2$, $x \neq 1$, and $x \neq 5$ from the divisor top. Answer $\dfrac{x-1}{x+2}$, $x \neq -2,\ 1,\ 5$.

   三个格子：$x \neq -2$，$x \neq 1$，以及除式分子给出的 $x \neq 5$。答案 $\dfrac{x-1}{x+2}$，$x \neq -2,\ 1,\ 5$。

2. **line · 第 1 —** In $x+5$ the x is a term that is ADDED, not a factor of the whole top. At $x=1$, $\frac{x+5}{x}=6$, not 5.

   $x+5$ 里的 x 是相加的项，不是整个分子的因式。$x=1$ 时 $\frac{x+5}{x}=6$，不是 5。

   Factor: $\dfrac{x+5}{x} \cdot \dfrac{2x}{(x-5)(x+5)}$; cancel x (top of one, bottom of the other) and $(x+5)$: $\dfrac{2}{x-5}$, $x \neq 0,\ \pm 5$.

   先分解：$\dfrac{x+5}{x} \cdot \dfrac{2x}{(x-5)(x+5)}$；约去 x（一个在分子、一个在另一分式的分母）和 $(x+5)$：$\dfrac{2}{x-5}$，$x \neq 0,\ \pm 5$。

3. **line · 第 2 —** Reading the NPVs from the final bottom misses $x+4$: it was a denominator in the ORIGINAL question and moved to the top when it was flipped. At $x=-4$ the original has $\frac{-3}{0}$.

   只看最后的分母会漏掉 $x+4$：它在「原题」里是分母，颠倒后才跑到分子上。$x=-4$ 时原式出现 $\frac{-3}{0}$。

   List NPVs before flipping: $x \neq 3$, $x \neq -4$, and $x \neq -1$ from the divisor top.

   颠倒之前列不允许值：$x \neq 3$，$x \neq -4$，以及除式分子给出的 $x \neq -1$。

4. **line · 第 2 —** $\dfrac{x^3}{x} = x^2$: one x cancels and two are left. Writing x subtracts 2 from the exponent instead of 1.

   $\dfrac{x^3}{x} = x^2$：约掉一个 x，还剩两个。写成 x 等于指数减了 2 而不是 1。

   $\dfrac{8x^3}{4x} = 2x^2$, so the answer is $2x^2(x-3)$, $x \neq 0,\ -3$.

   $\dfrac{8x^3}{4x} = 2x^2$，所以答案是 $2x^2(x-3)$，$x \neq 0,\ -3$。

5. **line · 第 1 —** Dividing by a fraction means multiplying by the reciprocal of the DIVISOR, the second fraction. Flipping the first one computes its upside-down value instead.

   除以一个分式等于乘以「除式」——也就是第二个分式——的倒数。颠倒第一个分式算的是完全不同的东西。

   $\dfrac{x}{x+3} \cdot \dfrac{(x-3)(x+3)}{x^2} = \dfrac{x-3}{x}$. NPVs: $x \neq -3$, $x \neq \pm 3$, and $x \neq 0$ from the divisor top x^2.

   $\dfrac{x}{x+3} \cdot \dfrac{(x-3)(x+3)}{x^2} = \dfrac{x-3}{x}$。不允许值：$x \neq -3$，$x \neq \pm 3$，以及除式分子 x^2 给出的 $x \neq 0$。