Difference of squares & perfect squares

平方差与完全平方

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Count the terms. Is there a common factor? Write each square term as $(\text{something})^2$, and note the sign between them.

   > 数一数有几项。有没有公因式？把每个平方项写成 $(某个式子)^2$，再看它们之间是什么符号。

2. 🎯 **Asked · 求什么** A product of brackets that cannot be split any further over the integers ("factor fully").

   > 写成乘积，而且每个括号在整数范围内都不能再分（「完全分解」）。

3. ✏️ **Picture it · 画出来** A big square of side A with a small square of side B cut from the corner: the leftover L-shape rearranges into an $(A-B)$ by $(A+B)$ rectangle. A trinomial $(A+B)^2$ is the big square split into A^2, B^2 and two AB strips.

   > 边长 A 的大正方形，从角上剪掉边长 B 的小正方形：剩下的 L 形能拼成 $(A-B)\times(A+B)$ 的长方形。三项式 $(A+B)^2$ 就是大正方形分成 A^2、B^2 和两条 AB。

4. 🧰 **Tool · 用什么工具** Common factor first. Two squares with a minus: $(A-B)(A+B)$. Three terms with square ends: test whether the middle is exactly $2AB$; if yes $(A\pm B)^2$, if not factor it as an ordinary trinomial.

   > 先提公因式。两个平方相减：$(A-B)(A+B)$。三项且首末是平方：检验中间项是不是正好 $2AB$；是就写 $(A\pm B)^2$，不是就按普通三项式分解。

5. ✅ **Check · 检验** Expand your answer and get the original back. Then look at every bracket once more: is any of them still a difference of squares?

   > 把答案展开，要能还原成原式。再把每个括号看一遍：还有没有平方差可以分？

⭐ Must know

必须记住

- $A^2 - B^2 = (A - B)(A + B)$: two squares with a MINUS.
  > $A^2 - B^2 = (A - B)(A + B)$：两个平方数，中间是减号。
- $A^2 \pm 2AB + B^2 = (A \pm B)^2$: check the middle is exactly $2AB$.
  > $A^2 \pm 2AB + B^2 = (A \pm B)^2$：一定要验证中间项正好是 $2AB$。
- $A^2 + B^2$ does not factor over the reals; keep going on $x^4 - 16$.
  > $A^2 + B^2$ 在实数范围内不能分解；$x^4 - 16$ 要继续分解。

$$A^2 - B^2 = (A - B)(A + B)$$

$$A^2 \pm 2AB + B^2 = (A \pm B)^2$$

🔍 Watch out for

当心

1. Factoring a sum of squares
   > 把平方和也分解了
2. Stopping before it is fully factored
   > 没分解完就停了
3. Writing a difference of squares as a perfect square
   > 把平方差写成了完全平方
4. Calling it a perfect square without checking the middle
   > 没验中间项就当成完全平方
5. Not looking for a common factor first
   > 没先找公因式

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can expand $(A + B)^2$ and $(A - B)(A + B)$, and I can see that 64 and $9x^4$ are perfect squares.
> 我会展开 $(A + B)^2$ 和 $(A - B)(A + B)$，也能看出 64、$9x^4$ 是完全平方。

(5′)

○ I can ask the five starting questions for a two-term or three-term expression.
> 我能对两项式或三项式问出五个入手问题。

(5′)

○ I can follow, then finish, a "common factor first, then difference of squares twice" factorization.
> 我能看懂并补全「先提公因式，再两次用平方差」的分解。

(8′)

○ I can factor a fourth-power expression fully on my own with at most two hints.
> 我能在最多两个提示下，独立把一个四次式完全分解。

(8′)

○ I can spot the five classic mistakes in someone else's factoring.
> 我能在别人的分解过程里找出五个经典错误。

(8′)

○ I can do the Level 1–2 practice questions.
> 我能做出 1–2 级练习题。

(10′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Expand $(2x+5)^2$.

> 展开 $(2x+5)^2$。

A. $4x^2+25$ B. $4x^2+10x+25$ C. $4x^2+20x+25$ D. $2x^2+20x+25$

If you missed it · 如果做错了 — **Multiplying binomials, $(a+b)^2$**

> **二项式相乘与 $(a+b)^2$**

: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

> 每一项乘每一项：$(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$。平方会有中间项：$(x+3)^2=x^2+6x+9$，绝不是 x^2+9。

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
   > $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$。
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.
   > 捷径：$(a\pm b)^2=a^2\pm 2ab+b^2$——中间项＝2×首项×末项。

2. Expand $(x-6)(x+6)$.

> 展开 $(x-6)(x+6)$。

A. x^2-36 B. x^2+36 C. $x^2-12x-36$ D. $x^2-6x+36$

If you missed it · 如果做错了 — **Distributing (and the minus sign)**

> **去括号（尤其是负号）**

: Multiply EVERY term inside the bracket: $3(x-4)=3x-12$. A minus in front changes every sign inside: $-(x-5)=-x+5$, and $2-(x+3)=2-x-3=-x-1$.

> 括号里的每一项都要乘到：$3(x-4)=3x-12$。前面是负号时，括号里每一项都变号：$-(x-5)=-x+5$，$2-(x+3)=2-x-3=-x-1$。

1. $-2(3x-1)=-6x+2$ (the -1 becomes $+2$).
   > $-2(3x-1)=-6x+2$（-1 变成 $+2$）。
2. $x(x+2)-3(x+2)=x^2+2x-3x-6=x^2-x-6$.
   > $x(x+2)-3(x+2)=x^2+2x-3x-6=x^2-x-6$。

3. Which number is a perfect square?

> 下面哪个数是完全平方数?

A. 18 B. 32 C. 50 D. 64

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1,4,9,\ldots,225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1,4,9,\ldots,225$ 这些平方数。$\sqrt{ab}=\sqrt{a}\sqrt{b}$，所以 $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$。但 $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$：$\sqrt{9+16}=5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2=49$ has TWO answers $x=\pm 7$, but $\sqrt{49}=7$ only.
   > $x^2=49$ 有两个解 $x=\pm 7$，但 $\sqrt{49}$ 只等于 7。

4. Simplify $(3x^2)^2$.

> 化简 $(3x^2)^2$。

A. $3x^4$ B. $9x^4$ C. $6x^4$ D. $9x^2$

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

> 同底数：相乘→指数相加（$x^3x^4 = x^7$）；相除→指数相减（$x^7 \div x^2 = x^5$）；幂的乘方→指数相乘（$(x^3)^2 = x^6$）。积的乘方每个因数都要乘方：$(2x)^3 = 8x^3$。

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
   > $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$。
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).
   > $\frac{6x^5}{2x^2} = 3x^3$（系数相除，指数相减）。

Answers · 答案: 1: C · 2: A · 3: D · 4: B

🔍 Common-errors sheet

常见错误单

1. Factoring a sum of squares

> **把平方和也分解了**

Factor fully: $16x^2 + 25$.

> 完全分解：$16x^2 + 25$。

1 Both terms are squares: $(4x)^2$ and 5^2.

> 两项都是平方：$(4x)^2$ 和 5^2。

2 So $16x^2 + 25 = (4x - 5)(4x + 5)$.

> 所以 $16x^2 + 25 = (4x - 5)(4x + 5)$。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Stopping before it is fully factored

> **没分解完就停了**

Factor fully: $16x^4 - 1$.

> 完全分解：$16x^4 - 1$。

1 $16x^4 = (4x^2)^2$ and $1 = 1^2$.

> $16x^4 = (4x^2)^2$，$1 = 1^2$。

2 $16x^4 - 1 = (4x^2 - 1)(4x^2 + 1)$. Done.

> $16x^4 - 1 = (4x^2 - 1)(4x^2 + 1)$。分解完了。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Writing a difference of squares as a perfect square

> **把平方差写成了完全平方**

Factor $9x^2 - 49$.

> 分解因式 $9x^2 - 49$。

1 $9x^2 = (3x)^2$ and $49 = 7^2$, with a minus between.

> $9x^2 = (3x)^2$，$49 = 7^2$，中间是减号。

2 So $9x^2 - 49 = (3x - 7)^2$.

> 所以 $9x^2 - 49 = (3x - 7)^2$。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Calling it a perfect square without checking the middle

> **没验中间项就当成完全平方**

Factor $x^2 + 13x + 36$.

> 分解因式 $x^2 + 13x + 36$。

1 $x^2 = (x)^2$ and $36 = 6^2$, so this is a perfect square.

> $x^2 = (x)^2$，$36 = 6^2$，所以是完全平方。

2 $x^2 + 13x + 36 = (x + 6)^2$.

> $x^2 + 13x + 36 = (x + 6)^2$。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Not looking for a common factor first

> **没先找公因式**

Factor fully: $12x^2 - 75$.

> 完全分解：$12x^2 - 75$。

1 12 is not a perfect square, and neither is 75.

> 12 不是完全平方数，75 也不是。

2 So $12x^2 - 75$ cannot be factored.

> 所以 $12x^2 - 75$ 不能分解。

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** The pattern $(A-B)(A+B)$ needs a MINUS between the squares. Expanding $(4x-5)(4x+5)$ gives $16x^2-25$, not $+25$.

   > 公式 $(A-B)(A+B)$ 要求两个平方之间是减号。展开 $(4x-5)(4x+5)$ 得 $16x^2-25$，不是 $+25$。

   A sum of two squares has no real factors, and 16 and 25 share no common factor, so it stays as it is.

   > 两个平方的和没有实数因式，16 和 25 也没有公因数，所以保持原样。

2. **line · 第 2 —** The first round is right, but "factor fully" means checking each bracket again. $4x^2-1=(2x)^2-1^2$ is another difference of squares.

   > 第一轮是对的，但「完全分解」要把每个括号再查一遍。$4x^2-1=(2x)^2-1^2$ 还是平方差。

   $4x^2-1=(2x-1)(2x+1)$; $4x^2+1$ is a sum and stays. So $(2x-1)(2x+1)(4x^2+1)$.

   > $4x^2-1=(2x-1)(2x+1)$；$4x^2+1$ 是平方和，保留。所以是 $(2x-1)(2x+1)(4x^2+1)$。

3. **line · 第 2 —** The minus sign got copied into both brackets. $(3x-7)^2=9x^2-42x+49$ has a middle term and ends in $+49$; a difference of squares needs one minus bracket and one plus bracket.

   > 把减号抄进了两个括号。$(3x-7)^2=9x^2-42x+49$ 有中间项，末项是 $+49$；平方差要一个减号括号、一个加号括号。

   $9x^2-49=(3x-7)(3x+7)$. Expand to check: $9x^2+21x-21x-49$ ✓.

   > $9x^2-49=(3x-7)(3x+7)$。展开检验：$9x^2+21x-21x-49$ ✓。

4. **line · 第 1 —** Square ends are not enough. For $(x+6)^2$ the middle would be $2\cdot x\cdot 6=12x$, but here it is $13x$.

   > 首末是平方还不够。$(x+6)^2$ 的中间项应是 $2\cdot x\cdot 6=12x$，这里却是 $13x$。

   Treat it as an ordinary trinomial: product 36, sum 13 gives 4 and 9, so $(x+4)(x+9)$.

   > 按普通三项式分解：积 36、和 13，得 4 和 9，所以是 $(x+4)(x+9)$。

5. **line · 第 2 —** The squares are hiding behind a common factor. Both 12 and 75 are multiples of 3, and after taking it out you get $4x^2-25$.

   > 平方藏在公因式后面。12 和 75 都是 3 的倍数，提出 3 后得到 $4x^2-25$。

   $12x^2-75=3(4x^2-25)=3(2x-5)(2x+5)$.

   > $12x^2-75=3(4x^2-25)=3(2x-5)(2x+5)$。