Factoring by substitution

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** Look for ONE chunk that shows up twice: once squared, once to the first power (x^4 and x^2, or $(x+2)^2$ and $(x+2)$). Write "Let $m =$" that chunk.
2. 🎯 **Asked** A full factorization in x, not in m, where no bracket can be factored again.
3. ✏️ **Picture it** Rewrite the whole expression with m, e.g. $m^2 - 5m + 4$. It should now look like an ordinary trinomial you already know how to factor.
4. 🧰 **Tool** Factor in m (product and sum, or decomposition when $a \neq 1$). Then put the chunk back in place of m and simplify each bracket.
5. ✅ **Check** Look at each bracket again: is it a difference of squares or does it have a common factor? Then put $x = 1$ (or 2) into the original and into your answer.

⭐ Must know

- Spot $a(\text{stuff})^2 + b(\text{stuff}) + c$; let $m =$ the stuff.
- Factor in m, substitute back, simplify each bracket.
- Re-check every bracket: $x^2 - 4$ and friends factor again.

$$a\,[f(x)]^2 + b\,f(x) + c \xrightarrow{m=f(x)} am^2 + bm + c$$

🔍 Watch out for

1. Leaving the answer in m
2. Stopping before the difference of squares
3. Putting x back instead of the whole chunk
4. Dropping the bracket when working out A minus B
5. Distributing only part of the way when substituting back

🌱 Small wins — tick as you go

- ○ I can factor $x^2 + bx + c$ and $x^2 - 25$, and subtract a whole bracket. (5′)
- ○ I can spot the repeated chunk and write "Let $m = \ldots$" as my first line. (4′)
- ○ I can follow, then finish, a substitute–factor–substitute back–factor again solution. (7′)
- ○ I can factor a quartic in quadratic form on my own with at most two hints. (5′)
- ○ I can spot the five classic quadratic-form mistakes in someone else's work. (5′)
- ○ I can do the Level 1–2 practice questions and check each by putting in $x = 1$. (4′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. Factor $x^2 - 7x + 12$.
A. $(x-3)(x-4)$ B. $(x+3)(x+4)$ C. $(x-2)(x-6)$ D. $(x-12)(x+1)$

If you missed it — **Factoring $x^2 + bx + c$ (product and sum)**: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: $x^2 - x - 12$ → -4 and 3 → $(x-4)(x+3)$. Check by expanding.

1. $x^2 + 7x + 10$: $2 \times 5 = 10$, $2 + 5 = 7$ → $(x+2)(x+5)$.
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.

2. Factor $x^2 - 25$.
A. $(x-5)^2$ B. $(x-5)(x+5)$ C. $(x+5)^2$ D. $x(x-25)$

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3. Simplify $(3x + 1) - (x - 4)$.
A. $2x - 3$ B. $2x + 5$ C. $4x - 3$ D. $2x + 3$

If you missed it — **Distributing (and the minus sign)**: Multiply EVERY term inside the bracket: $3(x-4) = 3x - 12$. A minus in front changes every sign inside: $-(x-5) = -x + 5$, and $2 - (x+3) = 2 - x - 3 = -x - 1$.

1. $-2(3x - 1) = -6x + 2$ (the -1 becomes $+2$).
2. $x(x+2) - 3(x+2) = x^2 + 2x - 3x - 6 = x^2 - x - 6$.

4. Which one equals x^6 written as something squared?
A. $(x^3)^2$ B. $(x^4)^2$ C. $(x^2)^4$ D. $(x^{36})^2$

If you missed it — **Exponent rules (product, quotient, power)**: Same base: multiply → ADD exponents ($x^3 x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).

Answers: 1: A · 2: B · 3: B · 4: A

🔍 Common-errors sheet

1. Leaving the answer in m

Factor fully: $x^4 - 6x^2 + 5$.

1 Let $m = x^2$: $m^2 - 6m + 5$

2 $= (m - 1)(m - 5)$

3 Answer: $(m - 1)(m - 5)$

✍ Which line goes wrong first, and why?

2. Stopping before the difference of squares

Factor fully: $x^4 - 20x^2 + 64$.

1 Let $m = x^2$: $(m - 4)(m - 16)$

2 Put x^2 back: $(x^2 - 4)(x^2 - 16)$

3 Done: $(x^2 - 4)(x^2 - 16)$

✍ Which line goes wrong first, and why?

3. Putting x back instead of the whole chunk

Factor $(x - 1)^2 + 3(x - 1) - 10$.

1 Let $m = x - 1$: $m^2 + 3m - 10 = (m + 5)(m - 2)$

2 Replace m with x: $(x + 5)(x - 2)$

✍ Which line goes wrong first, and why?

4. Dropping the bracket when working out A minus B

Factor fully: $(3x-1)^2-(x+5)^2$.

1 $A-B=3x-1-x+5=2x+4$

2 $A+B=4x+4$

3 $(2x+4)(4x+4)$

✍ Which line goes wrong first, and why?

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5. Distributing only part of the way when substituting back

Factor $3(x-2)^2+7(x-2)+2$.

1 Let $m=x-2$: $3m^2+7m+2=(3m+1)(m+2)$

2 Back: $(3x-2+1)(x-2+2)$

3 $=x(3x-1)$

✍ Which line goes wrong first, and why?

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Key (fold this part back)

1. **line 3 —** The factoring felt like the finish line. But m was only a helper letter; the question is about x. Put x^2 back: $(x^2-1)(x^2-5)=(x-1)(x+1)(x^2-5)$. ($x^2-5$ does not factor over the integers.)
2. **line 3 —** After the substitution the work looks finished. But "fully" means checking every bracket again, and x^2-4, x^2-16 are both differences of squares. $(x-2)(x+2)(x-4)(x+4)$.
3. **line 2 —** In the quartic questions m was "x-something" and it is easy to just write x. Here m stands for the whole $x-1$. $\big((x-1)+5\big)\big((x-1)-2\big)=(x+4)(x-3)$.
4. **line 1 —** The minus sign was given to x only. B is the whole $(x+5)$, so the minus hits the 5 too. $A-B=(3x-1)-(x+5)=2x-6$, $A+B=4x+4$, so $(2x-6)(4x+4)=8(x-3)(x+1)$.
5. **line 2 —** $3m$ became $3x-2$ because the bracket was not written. $3m=3(x-2)=3x-6$. $\big(3(x-2)+1\big)\big((x-2)+2\big)=(3x-5)x$, so $x(3x-5)$.