Factoring by substitution

换元法分解

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Look for ONE chunk that shows up twice: once squared, once to the first power (x^4 and x^2, or $(x+2)^2$ and $(x+2)$). Write "Let $m =$" that chunk.
   > 找出出现了两次的「同一块」：一次是平方，一次是一次方（x^4 和 x^2，或 $(x+2)^2$ 和 $(x+2)$）。写下「设 $m =$」这一块。
2. 🎯 **Asked · 求什么** A full factorization in x, not in m, where no bracket can be factored again.
   > 用 x 写的完全分解（不是用 m），每个括号都不能再分解。
3. ✏️ **Picture it · 画出来** Rewrite the whole expression with m, e.g. $m^2 - 5m + 4$. It should now look like an ordinary trinomial you already know how to factor.
   > 用 m 把整个式子重写一遍，例如 $m^2 - 5m + 4$。它应该变成一个你已经会分解的普通三项式。
4. 🧰 **Tool · 用什么工具** Factor in m (product and sum, or decomposition when $a \neq 1$). Then put the chunk back in place of m and simplify each bracket.
   > 按 m 分解（积与和；$a \neq 1$ 时用拆项法）。再把这一块代回 m 的位置，化简每个括号。
5. ✅ **Check · 检验** Look at each bracket again: is it a difference of squares or does it have a common factor? Then put $x = 1$ (or 2) into the original and into your answer.
   > 再看一遍每个括号：是不是平方差？有没有公因数？然后把 $x = 1$（或 2）分别代入原式和答案。

⭐ Must know

必须记住

- Spot $a(\text{stuff})^2 + b(\text{stuff}) + c$; let $m =$ the stuff.
  > 认出 $a(\text{整体})^2 + b(\text{整体}) + c$；令 $m =$ 这个整体。
- Factor in m, substitute back, simplify each bracket.
  > 按 m 分解，代回，再化简每个括号。
- Re-check every bracket: $x^2 - 4$ and friends factor again.
  > 每个括号再查一遍：$x^2 - 4$ 这类还能继续分解。

$$a\,[f(x)]^2 + b\,f(x) + c \xrightarrow{m=f(x)} am^2 + bm + c$$

🔍 Watch out for

当心

1. Leaving the answer in m
   > 答案停在 m 上
2. Stopping before the difference of squares
   > 没分解平方差就停了
3. Putting x back instead of the whole chunk
   > 代回时只写了 x，没代回整块
4. Dropping the bracket when working out A minus B
   > 算 A−B 时丢了括号
5. Distributing only part of the way when substituting back
   > 代回时分配律只做了一半

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can factor $x^2 + bx + c$ and $x^2 - 25$, and subtract a whole bracket.

> 我会分解 $x^2 + bx + c$ 和 $x^2 - 25$，也会减去一整个括号。

(5′)

○ I can spot the repeated chunk and write "Let $m = \ldots$" as my first line.

> 我能认出重复出现的那一块，第一行就写「设 $m = \ldots$」。

(4′)

○ I can follow, then finish, a substitute–factor–substitute back–factor again solution.

> 我能看懂并补全「换元—分解—代回—再分解」的解法。

(7′)

○ I can factor a quartic in quadratic form on my own with at most two hints.

> 我能在最多两个提示下，独立分解一个二次型的四次式。

(5′)

○ I can spot the five classic quadratic-form mistakes in someone else's work.

> 我能在别人的解答里找出二次型分解的五个经典错误。

(5′)

○ I can do the Level 1–2 practice questions and check each by putting in $x = 1$.

> 我能做出 1–2 级练习题，并代 $x = 1$ 检验每一道。

(4′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Factor $x^2 - 7x + 12$.

> 分解因式 $x^2 - 7x + 12$。

A. $(x-3)(x-4)$ B. $(x+3)(x+4)$ C. $(x-2)(x-6)$ D. $(x-12)(x+1)$

If you missed it · 如果做错了 — **Factoring $x^2 + bx + c$ (product and sum)**

> **分解 $x^2 + bx + c$（积与和）**

: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: $x^2 - x - 12$ → -4 and 3 → $(x-4)(x+3)$. Check by expanding.

> 先提公因式。再找两个数，相乘等于 c、相加等于 b：$x^2 - x - 12$ → -4 和 3 → $(x-4)(x+3)$。展开检验。

1. $x^2 + 7x + 10$: $2 \times 5 = 10$, $2 + 5 = 7$ → $(x+2)(x+5)$.
   > $x^2 + 7x + 10$：$2 \times 5 = 10$，$2 + 5 = 7$ → $(x+2)(x+5)$。
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.
   > 若 $c < 0$，两个数异号；若 $c > 0$，两个数与 b 同号。

2. Factor $x^2 - 25$.

> 分解因式 $x^2 - 25$。

A. $(x-5)^2$ B. $(x-5)(x+5)$ C. $(x+5)^2$ D. $x(x-25)$

If you missed it · 如果做错了 — **Factoring $x^2 + bx + c$ (product and sum)**

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   > $x^2 + 7x + 10$：$2 \times 5 = 10$，$2 + 5 = 7$ → $(x+2)(x+5)$。
2. If $c < 0$ the two numbers have different signs; if $c > 0$ they share the sign of b.
   > 若 $c < 0$，两个数异号；若 $c > 0$，两个数与 b 同号。

3. Simplify $(3x + 1) - (x - 4)$.

> 化简 $(3x + 1) - (x - 4)$。

A. $2x - 3$ B. $2x + 5$ C. $4x - 3$ D. $2x + 3$

If you missed it · 如果做错了 — **Distributing (and the minus sign)**

> **去括号（尤其是负号）**

: Multiply EVERY term inside the bracket: $3(x-4) = 3x - 12$. A minus in front changes every sign inside: $-(x-5) = -x + 5$, and $2 - (x+3) = 2 - x - 3 = -x - 1$.

> 括号里的每一项都要乘到：$3(x-4) = 3x - 12$。前面是负号时，括号里每一项都变号：$-(x-5) = -x + 5$，$2 - (x+3) = 2 - x - 3 = -x - 1$。

1. $-2(3x - 1) = -6x + 2$ (the -1 becomes $+2$).
   > $-2(3x - 1) = -6x + 2$（-1 变成 $+2$）。
2. $x(x+2) - 3(x+2) = x^2 + 2x - 3x - 6 = x^2 - x - 6$.
   > $x(x+2) - 3(x+2) = x^2 + 2x - 3x - 6 = x^2 - x - 6$。

4. Which one equals x^6 written as something squared?

> 把 x^6 写成「某个式子的平方」，是哪一个?

A. $(x^3)^2$ B. $(x^4)^2$ C. $(x^2)^4$ D. $(x^{36})^2$

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

> 同底数：相乘→指数相加（$x^3x^4 = x^7$）；相除→指数相减（$x^7 \div x^2 = x^5$）；幂的乘方→指数相乘（$(x^3)^2 = x^6$）。积的乘方每个因数都要乘方：$(2x)^3 = 8x^3$。

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
   > $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$。
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).
   > $\frac{6x^5}{2x^2} = 3x^3$（系数相除，指数相减）。

Answers · 答案: 1: A · 2: B · 3: B · 4: A

🔍 Common-errors sheet

常见错误单

1. Leaving the answer in m

答案停在 m 上

Factor fully: $x^4 - 6x^2 + 5$.

完全分解：$x^4 - 6x^2 + 5$。

1 Let $m = x^2$: $m^2 - 6m + 5$

设 $m = x^2$：$m^2 - 6m + 5$

2 $= (m - 1)(m - 5)$

3 Answer: $(m - 1)(m - 5)$

答案：$(m - 1)(m - 5)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Stopping before the difference of squares

没分解平方差就停了

Factor fully: $x^4 - 20x^2 + 64$.

完全分解：$x^4 - 20x^2 + 64$。

1 Let $m = x^2$: $(m - 4)(m - 16)$

设 $m = x^2$：$(m - 4)(m - 16)$

2 Put x^2 back: $(x^2 - 4)(x^2 - 16)$

代回 x^2：$(x^2 - 4)(x^2 - 16)$

3 Done: $(x^2 - 4)(x^2 - 16)$

完成：$(x^2 - 4)(x^2 - 16)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Putting x back instead of the whole chunk

> **代回时只写了 x，没代回整块**

Factor $(x-1)^2+3(x-1)-10$.

> 分解因式 $(x-1)^2+3(x-1)-10$。

1 Let $m=x-1$: $m^2+3m-10=(m+5)(m-2)$

> 设 $m=x-1$：$m^2+3m-10=(m+5)(m-2)$

2 Replace m with x: $(x+5)(x-2)$

> 把 m 换成 x：$(x+5)(x-2)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Dropping the bracket when working out A minus B

> **算 A−B 时丢了括号**

Factor fully: $(3x-1)^2-(x+5)^2$.

> 完全分解：$(3x-1)^2-(x+5)^2$。

1 $A-B=3x-1-x+5=2x+4$

2 $A+B=4x+4$

3 $(2x+4)(4x+4)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Distributing only part of the way when substituting back

> **代回时分配律只做了一半**

Factor $3(x-2)^2+7(x-2)+2$.

> 分解因式 $3(x-2)^2+7(x-2)+2$。

1 Let $m=x-2$: $3m^2+7m+2=(3m+1)(m+2)$

> 设 $m=x-2$：$3m^2+7m+2=(3m+1)(m+2)$

2 Back: $(3x-2+1)(x-2+2)$

> 代回：$(3x-2+1)(x-2+2)$

3 $=x(3x-1)$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 3 —** The factoring felt like the finish line. But m was only a helper letter; the question is about x.

   > 分解完就觉得到终点了。可 m 只是临时的帮手，题目问的是 x。

   Put x^2 back: $(x^2-1)(x^2-5) = (x-1)(x+1)(x^2-5)$. ($x^2-5$ does not factor over the integers.)

   > 把 x^2 代回：$(x^2-1)(x^2-5) = (x-1)(x+1)(x^2-5)$。（$x^2-5$ 在整数范围内不能再分。）

2. **line · 第 3 —** After the substitution the work looks finished. But "fully" means checking every bracket again, and x^2-4, x^2-16 are both differences of squares.

   > 换元做完，看起来就结束了。可是「完全」分解要求每个括号再查一遍，x^2-4 和 x^2-16 都是平方差。

   $(x-2)(x+2)(x-4)(x+4)$.

   > $(x-2)(x+2)(x-4)(x+4)$。

3. **line · 第 2 —** In the quartic questions m was "x-something" and it is easy to just write x. Here m stands for the whole $x-1$.

   > 做四次式时习惯把 m 当成「和 x 差不多」，顺手就写成了 x。这里的 m 代表整个 $x-1$。

   $\big((x-1)+5\big)\big((x-1)-2\big) = (x+4)(x-3)$.

   > $\big((x-1)+5\big)\big((x-1)-2\big) = (x+4)(x-3)$。

4. **line · 第 1 —** The minus sign was given to x only. B is the whole $(x+5)$, so the minus hits the 5 too.

   > 减号只给了 x。B 是整个 $(x+5)$，所以减号也要作用到 5。

   $A-B = (3x-1)-(x+5) = 2x-6$, $A+B = 4x+4$, so $(2x-6)(4x+4) = 8(x-3)(x+1)$.

   > $A-B = (3x-1)-(x+5) = 2x-6$，$A+B = 4x+4$，所以 $(2x-6)(4x+4) = 8(x-3)(x+1)$。

5. **line · 第 2 —** $3m$ became $3x-2$ because the bracket was not written. $3m = 3(x-2) = 3x-6$.

   > 没写括号，$3m$ 就成了 $3x-2$。其实 $3m = 3(x-2) = 3x-6$。

   $\big(3(x-2)+1\big)\big((x-2)+2\big) = (3x-5)x$, so $x(3x-5)$.

   > $\big(3(x-2)+1\big)\big((x-2)+2\big) = (3x-5)x$，即 $x(3x-5)$。