Simplifying & ordering radicals

根式化简与比较大小

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Read the index (2 or 3) and the radicand. With letters, note which exponents are odd and whether you are told $x \geq 0$.
   > 读出根指数（2 还是 3）和被开方数。有字母时，看哪些指数是奇数，题目有没有说 $x \geq 0$。
2. 🎯 **Asked · 求什么** Simplest form: no perfect-square factor left inside (perfect cube for $\sqrt[3]{}$), plus a restriction if an even root holds a letter. For ordering: a list from least to greatest.
   > 最简形式：根号里不再有完全平方因数（立方根就是完全立方因数）；偶次根号里有字母时还要写限制条件。排大小时：从小到大的一串。
3. ✏️ **Picture it · 画出来** A factor tree of the radicand. Circle each PAIR of equal primes (each triple for a cube root); every circled group comes out as one number.
   > 给被开方数画因数树。把相同质数「两个一圈」（立方根就三个一圈）；每圈出一组，就有一个数出到根号外。
4. 🧰 **Tool · 用什么工具** Try the perfect squares from big to small (100, 81, 64, 49, 36, 25, 16, 9, 4) until one divides the radicand; split and take its root out. To compare, square the outside number and move it in.
   > 从大到小试完全平方数（100、81、64、49、36、25、16、9、4），直到有一个能整除被开方数；拆开，把它的平方根提出去。比较大小时，把外面的数平方后乘进去。
5. ✅ **Check · 检验** Multiply back: (outside)$^2\times$ inside must give the original radicand. Then ask: can the leftover still be divided by 4, 9 or 25?
   > 乘回去：（外面的数）$^2\times$ 里面的数必须等于原来的被开方数。再问：剩下的还能被 4、9 或 25 整除吗?

⭐ Must know

必须记住

- Pull out the LARGEST perfect square (perfect cube for $\sqrt[3]{}$): $\sqrt{72} = 6\sqrt{2}$.
  > 提出最大的完全平方因数（立方根提完全立方数）：$\sqrt{72} = 6\sqrt{2}$。
- Mixed → entire: square the outside number and multiply it in: $3\sqrt{5} = \sqrt{45}$.
  > 最简根式化为整根式：把外面的数平方后乘进去：$3\sqrt{5} = \sqrt{45}$。
- To compare radicals, rewrite all as entire radicals and compare the insides.
  > 比较根式大小：都化成整根式，比较根号里面的数。

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b} \quad (a, b \geq 0)$$

$$k\sqrt[n]{a} = \sqrt[n]{k^n a}$$

🔍 Watch out for

当心

1. Stopping before the largest square is out
   > 没提出最大的平方因数就停了
2. Not squaring the outside number when moving it in
   > 把外面的数移进根号时没有平方
3. Comparing only the outside numbers
   > 比较大小只看外面的数
4. Taking a whole odd power out of the root
   > 把奇次幂整个提出根号
5. Splitting a root over a plus sign
   > 把根号对加法拆开

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can name the perfect squares up to 144, find $\sqrt[3]{-27}$, and split a number into primes.
  > 我能说出 144 以内的完全平方数，会求 $\sqrt[3]{-27}$，会把一个数分解质因数。

  (5′)
- ○ I can read the index and the radicand and say what "simplest form" has to look like.
  > 我能读出根指数和被开方数，并说出「最简形式」应该是什么样子。

  (5′)
- ○ I can follow, then finish, a largest-perfect-square simplification and check it by multiplying back.
  > 我能看懂并补全「找最大完全平方因数」的化简，并用乘回去的方法检验。

  (8′)
- ○ I can simplify one on my own with at most two hints.
  > 我能在最多两个提示下独立化简一道。

  (6′)
- ○ I can spot the five classic radical mistakes in someone else's work.
  > 我能在别人的解答里找出根式化简的五个经典错误。

  (6′)
- ○ I can do the Level 1–2 practice questions, including putting radicals in order.
  > 我能做出 1–2 级练习题，包括给根式排大小。

  (5′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Which number is a perfect square?

> 哪个数是完全平方数?

A. 32 B. 48 C. 64 D. 72

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**
> : Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.
>
> 记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

2. What is $\sqrt[3]{-27}$?

> $\sqrt[3]{-27}$ 等于多少?

A. -3 B. 3 C. -9 D. It has no real value

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3. Write 72 as a product of prime numbers.

> 把 72 写成质数的乘积。

A. $2^2 \cdot 3^3$ B. $2^3 \cdot 3^2$ C. $2 \cdot 36$ D. $2^4 \cdot 3$

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

> 同底数：相乘→指数相加（$x^3x^4 = x^7$）；相除→指数相减（$x^7 \div x^2 = x^5$）；幂的乘方→指数相乘（$(x^3)^2 = x^6$）。积的乘方每个因数都要乘方：$(2x)^3 = 8x^3$。

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
   > $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$。
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).
   > $\frac{6x^5}{2x^2} = 3x^3$（系数相除，指数相减）。

4. Simplify $\sqrt{x^6}$ for $x \geq 0$.

> 当 $x \geq 0$ 时，化简 $\sqrt{x^6}$。

A. x^4 B. $3x$ C. x^{36} D. x^3

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Answers · 答案: 1: C · 2: A · 3: B · 4: D

🔍 Common-errors sheet

常见错误单

1. Stopping before the largest square is out

没提出最大的平方因数就停了

Simplify $\sqrt{200}$.

化简 $\sqrt{200}$。

1 $200 = 4 \cdot 50$, so $\sqrt{200} = 2\sqrt{50}$

$200 = 4 \cdot 50$，所以 $\sqrt{200} = 2\sqrt{50}$

2 Final answer: $2\sqrt{50}$

最终答案：$2\sqrt{50}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Not squaring the outside number when moving it in

把外面的数移进根号时没有平方

Write $5\sqrt{3}$ as an entire radical.

把 $5\sqrt{3}$ 写成整根式。

1 $5\sqrt{3} = \sqrt{5 \cdot 3}$

2 $= \sqrt{15}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Comparing only the outside numbers

比较大小只看外面的数

Which is greater, $3\sqrt{7}$ or $2\sqrt{17}$?

$3\sqrt{7}$ 和 $2\sqrt{17}$ 哪个大?

1 Compare the outside numbers: $3 > 2$.

比较外面的数：$3 > 2$。

2 So $3\sqrt{7} > 2\sqrt{17}$.

所以 $3\sqrt{7} > 2\sqrt{17}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Taking a whole odd power out of the root

把奇次幂整个提出根号

Simplify $\sqrt{20x^5}$ and state the restriction.

化简 $\sqrt{20x^5}$，并写出限制条件。

1 $20 = 4 \cdot 5$, so $\sqrt{20} = 2\sqrt{5}$

$20 = 4 \cdot 5$，所以 $\sqrt{20} = 2\sqrt{5}$

2 $\sqrt{x^5} = x^5$

3 Answer: $2x^5\sqrt{5}$

答案：$2x^5\sqrt{5}$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Splitting a root over a plus sign

把根号对加法拆开

Simplify $\sqrt{36 + 64}$.

化简 $\sqrt{36 + 64}$。

1 $\sqrt{36 + 64} = \sqrt{36} + \sqrt{64}$

2 $= 6 + 8 = 14$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** The first square that came to mind was 4, and line 1 is true. But $50 = 25 \cdot 2$ still hides a square, so the answer is not finished.

   > 最先想到的平方数是 4，第 1 行本身没错。可 $50 = 25 \cdot 2$ 里还藏着平方数，所以还没化完。

   Keep going: $2\sqrt{50} = 2 \cdot 5\sqrt{2} = 10\sqrt{2}$. Faster: $200 = 100 \cdot 2$.

   > 继续：$2\sqrt{50} = 2 \cdot 5\sqrt{2} = 10\sqrt{2}$。更快的做法：$200 = 100 \cdot 2$。

2. **line · 第 1 —** Moving in feels like "just put it under the root". But 5 is the root of 25, so it goes in as 25.

   > 感觉「移进去」就是直接放进根号。可是 5 是 25 的平方根，所以进去的是 25。

   $5\sqrt{3} = \sqrt{25 \cdot 3} = \sqrt{75}$. Check: $\sqrt{15} \approx 3.9$ but $5\sqrt{3} \approx 8.7$.

   > $5\sqrt{3} = \sqrt{25 \cdot 3} = \sqrt{75}$。检验：$\sqrt{15} \approx 3.9$，而 $5\sqrt{3} \approx 8.7$。

3. **line · 第 1 —** It looks like comparing coefficients of like terms. But the radicands are different, so the outside number alone tells you nothing.

   > 看起来像比较同类项的系数。可是被开方数不同，只看外面的数说明不了什么。

   Move the outside numbers in: $3\sqrt{7} = \sqrt{63}$ and $2\sqrt{17} = \sqrt{68}$. Since $68 > 63$, $2\sqrt{17}$ is greater.

   > 把外面的数移进去：$3\sqrt{7} = \sqrt{63}$，$2\sqrt{17} = \sqrt{68}$。因为 $68 > 63$，所以 $2\sqrt{17}$ 更大。

4. **line · 第 2 —** The letter was treated as if it were already a square. A square root HALVES an exponent, and only an even power can come out: $x^5 = x^4 \cdot x$.

   > 把字母当成已经是平方了。平方根会让指数「减半」，只有偶次幂才能整个出来：$x^5 = x^4 \cdot x$。

   $20x^5 = (4x^4)(5x)$, so $\sqrt{20x^5} = 2x^2\sqrt{5x}$, with $x \geq 0$.

   > $20x^5 = (4x^4)(5x)$，所以 $\sqrt{20x^5} = 2x^2\sqrt{5x}$，且 $x \geq 0$。

5. **line · 第 1 —** The product rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is real, so it is tempting to use the same move for $+$. There is no such rule for addition.

   > 乘法法则 $\sqrt{ab} = \sqrt{a}\sqrt{b}$ 是对的，于是很想对 $+$ 也这样拆。加法没有这条法则。

   Add first: $\sqrt{100} = 10$. (Check: $14^2 = 196 \neq 100$.)

   > 先加：$\sqrt{100} = 10$。（检验：$14^2 = 196 \neq 100$。）