Operations with radicals

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** Which operation is it: add/subtract, multiply, or a radical in the denominator? Simplify every radical first; unlike-looking ones often turn out alike.
2. 🎯 **Asked** An exact answer in simplest form: like radicals collected, no square factor inside a root, and no root left in the denominator.
3. ✏️ **Picture it** For two brackets, a 2×2 box (rows = first bracket, columns = second). For a binomial denominator, write its conjugate right under it with the middle sign flipped.
4. 🧰 **Tool** Add only like radicals. Multiply outside × outside, inside × inside. One-term denominator: multiply by $\frac{\sqrt{b}}{\sqrt{b}}$. Two-term denominator: multiply top AND bottom by the conjugate.
5. ✅ **Check** Type the original and your answer into a calculator; they must agree to at least 3 decimal places.

⭐ Must know

- Add/subtract LIKE radicals only — simplify each first: $\sqrt{12}+\sqrt{27}=5\sqrt{3}$.
- Multiply outside × outside, inside × inside; expand brackets like FOIL.
- Rationalize: $\times\frac{\sqrt{b}}{\sqrt{b}}$; binomial bottom → × conjugate, $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$.

$$m\sqrt[n]{a}\cdot k\sqrt[n]{b}=mk\sqrt[n]{ab}$$

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b$$

🔍 Watch out for

1. Adding radicals by adding the insides
2. Squaring a binomial without the middle term
3. Multiplying only the bottom by the conjugate
4. Forgetting to square the number in front of the root
5. Dividing only one term of the top

🌱 Small wins — tick as you go

- ○ I can expand $(x+3)(x-3)$ and $(2x-1)^2$, simplify $\sqrt{48}$, and divide every term of a sum. (6′)
- ○ I can say which operation a question is (add, multiply, or rationalize) and what the final form must look like. (5′)
- ○ I can follow, then finish, a conjugate rationalization and check it with decimals. (10′)
- ○ I can rationalize a binomial denominator on my own with at most two hints. (8′)
- ○ I can spot the five classic radical-operation mistakes in someone else's work. (8′)
- ○ I can do the Level 1–2 practice questions and decimal-check each answer. (8′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. Expand $(x+3)(x-3)$.
A. x^2-9 B. x^2+9 C. x^2-6x-9 D. x^2-6x+9

If you missed it — **Multiplying binomials,** $(a+b)^2$: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.

2. Expand $(2x-1)^2$.
A. $4x^2+1$ B. $4x^2-1$ C. $4x^2-4x+1$ D. $2x^2-4x+1$

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3. Simplify $\sqrt{48}$.
A. $2\sqrt{12}$ B. $4\sqrt{3}$ C. $16\sqrt{3}$ D. $3\sqrt{4}$

If you missed it — **Perfect squares and square roots**: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
2. $x^2=49$ has TWO answers $x=\pm 7$, but $\sqrt{49}=7$ only.

4. Simplify $\dfrac{6x+9}{3}$.
A. $2x+9$ B. $2x+3$ C. $6x+3$ D. $3x+3$

If you missed it — **Fractions: add, subtract, multiply, divide**: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.

Answers: 1: A · 2: C · 3: B · 4: B

🔍 Common-errors sheet

1. Adding radicals by adding the insides

Simplify $\sqrt{8}+\sqrt{18}$.

1 $\sqrt{8}+\sqrt{18}=\sqrt{26}$

2 $26=2\cdot 13$ has no square factor, so the answer is $\sqrt{26}$.

✍ Which line goes wrong first, and why?

2. Squaring a binomial without the middle term

Expand $(\sqrt{5}-2)^2$.

1 $(\sqrt{5}-2)^2=(\sqrt{5})^2+(-2)^2$

2 $=5+4=9$

✍ Which line goes wrong first, and why?

3. Multiplying only the bottom by the conjugate

Rationalize $\dfrac{2}{\sqrt{6}+2}$.

1 Multiply the bottom by $\sqrt{6}-2$: $(\sqrt{6}+2)(\sqrt{6}-2)=6-4=2$

2 So the fraction is $\dfrac{2}{2}=1$.

✍ Which line goes wrong first, and why?

4. Forgetting to square the number in front of the root

Rationalize $\dfrac{14}{3\sqrt{2}-2}$.

1 Multiply top and bottom by $3\sqrt{2}+2$.

2 Bottom: $(3\sqrt{2})^2 - 2^2 = 6 - 4 = 2$

3 $\dfrac{14(3\sqrt{2}+2)}{2} = 21\sqrt{2}+14$

✍ Which line goes wrong first, and why?

5. Dividing only one term of the top

Simplify $\dfrac{12+4\sqrt{5}}{4}$.

1 $12 \div 4 = 3$

2 So the answer is $3+4\sqrt{5}$.

✍ Which line goes wrong first, and why?

Key (fold this part back)

1. **line 1** — Adding the insides feels like the same move as multiplying them. Only like radicals add, and only their outside numbers: like $2a+3a=5a$. Simplify first: $2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$. Decimal check: $2.83+4.24=7.07=5\sqrt{2}$, not $\sqrt{26}\approx 5.10$.
2. **line 1** — Squaring each piece feels like the power rule $(ab)^2=a^2b^2$, but that works for products, not sums. $(A-B)^2$ means $(A-B)(A-B)$: four products. $5-2\cdot 2\sqrt{5}+4=9-4\sqrt{5}$.
3. **line 1** — The focus is on getting rid of the root below, so the top is forgotten. Multiplying only the bottom changes the value; top and bottom together is multiplying by 1. $\dfrac{2(\sqrt{6}-2)}{2}=\sqrt{6}-2\approx 0.449$. Check: $2\div 4.449\approx 0.449$, not 1.
4. **line 2** — $(3\sqrt{2})^2$ was read as "3 times 2". The 3 is squared too: $(3\sqrt{2})^2=9\cdot 2=18$. Bottom: $18-4=14$, so $\dfrac{14(3\sqrt{2}+2)}{14}=3\sqrt{2}+2$.
5. **line 2** — The 4 below divides the WHOLE top. Cancelling with the first number only quietly turns the question into $\dfrac{12}{4}+4\sqrt{5}$. Divide both terms: $\dfrac{12}{4}+\dfrac{4\sqrt{5}}{4}=3+\sqrt{5}$.