Operations with radicals

根式的四则运算

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Which operation is it: add/subtract, multiply, or a radical in the denominator? Simplify every radical first; unlike-looking ones often turn out alike.

   > 是哪种运算：加减、相乘，还是分母里有根号？先把每个根式化简；看起来不同类的常常会变成同类。

2. 🎯 **Asked · 求什么** An exact answer in simplest form: like radicals collected, no square factor inside a root, and no root left in the denominator.

   > 最简精确值：同类根式已合并，根号里没有平方因数，分母里不留根号。

3. ✏️ **Picture it · 画出来** For two brackets, a 2×2 box (rows = first bracket, columns = second). For a binomial denominator, write its conjugate right under it with the middle sign flipped.

   > 两个括号相乘时，画一个 2×2 方格（行是第一个括号，列是第二个）。分母是二项式时，在它正下方写出共轭式，中间的符号反过来。

4. 🧰 **Tool · 用什么工具** Add only like radicals. Multiply outside × outside, inside × inside. One-term denominator: multiply by $\frac{\sqrt{b}}{\sqrt{b}}$. Two-term denominator: multiply top AND bottom by the conjugate.

   > 只合并同类根式。相乘：外乘外、里乘里。分母是单项：乘 $\frac{\sqrt{b}}{\sqrt{b}}$。分母是两项：分子分母「同时」乘共轭式。

5. ✅ **Check · 检验** Type the original and your answer into a calculator; they must agree to at least 3 decimal places.

   > 把原式和你的答案都输进计算器，至少小数点后 3 位要一致。

⭐ Must know

必须记住

- Add/subtract LIKE radicals only — simplify each first: $\sqrt{12}+\sqrt{27}=5\sqrt{3}$.
  > 只有同类根式才能加减——先逐个化简：$\sqrt{12}+\sqrt{27}=5\sqrt{3}$。
- Multiply outside × outside, inside × inside; expand brackets like FOIL.
  > 相乘：系数乘系数，根号里乘根号里；括号像多项式一样展开。
- Rationalize: $\times\frac{\sqrt{b}}{\sqrt{b}}$; binomial bottom → × conjugate, $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$.
  > 分母有理化：单项乘 $\frac{\sqrt{b}}{\sqrt{b}}$；二项分母乘共轭式，$(a+\sqrt{b})(a-\sqrt{b})=a^2-b$。

$$m\sqrt[n]{a}\cdot k\sqrt[n]{b}=mk\sqrt[n]{ab}$$

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b$$

🔍 Watch out for

当心

1. Adding radicals by adding the insides
   > 根式相加时把根号里的数相加
2. Squaring a binomial without the middle term
   > 二项式平方漏了中间项
3. Multiplying only the bottom by the conjugate
   > 只给分母乘共轭式
4. Forgetting to square the number in front of the root
   > 忘了把根号前的系数平方
5. Dividing only one term of the top
   > 约分时只除了分子的一项

🌱 Small wins — tick as you go

一小步一小步——做完打勾

○ I can expand $(x+3)(x-3)$ and $(2x-1)^2$, simplify $\sqrt{48}$, and divide every term of a sum.
> 我会展开 $(x+3)(x-3)$ 和 $(2x-1)^2$，会化简 $\sqrt{48}$，也会把和的每一项都除掉。
(6′)

○ I can say which operation a question is (add, multiply, or rationalize) and what the final form must look like.
> 我能说出题目是哪种运算（加减、乘、还是有理化），以及最终答案应该是什么样。
(5′)

○ I can follow, then finish, a conjugate rationalization and check it with decimals.
> 我能看懂并补全一道乘共轭式的有理化，并用小数检验。
(10′)

○ I can rationalize a binomial denominator on my own with at most two hints.
> 我能在最多两个提示下，独立把二项式分母有理化。
(8′)

○ I can spot the five classic radical-operation mistakes in someone else's work.
> 我能在别人的解答里找出根式运算的五个经典错误。
(8′)

○ I can do the Level 1–2 practice questions and decimal-check each answer.
> 我能做出 1–2 级练习题，并用小数检验每个答案。
(8′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Expand $(x+3)(x-3)$.

展开 $(x+3)(x-3)$。

A. x^2-9 B. x^2+9 C. x^2-6x-9 D. x^2-6x+9

If you missed it · 如果做错了 — **Multiplying binomials, $(a+b)^2$**

二项式相乘与 $(a+b)^2$

: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

每一项乘每一项：$(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$。平方会有中间项：$(x+3)^2=x^2+6x+9$，绝不是 x^2+9。

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
   $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$。
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.
   捷径：$(a\pm b)^2=a^2\pm 2ab+b^2$——中间项=2×首项×末项。

2. Expand $(2x-1)^2$.

展开 $(2x-1)^2$。

A. $4x^2+1$ B. $4x^2-1$ C. $4x^2-4x+1$ D. $2x^2-4x+1$

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每一项乘每一项：$(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$。平方会有中间项：$(x+3)^2=x^2+6x+9$，绝不是 x^2+9。

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   捷径：$(a\pm b)^2=a^2\pm 2ab+b^2$——中间项=2×首项×末项。

3. Simplify $\sqrt{48}$.

化简 $\sqrt{48}$。

A. $2\sqrt{12}$ B. $4\sqrt{3}$ C. $16\sqrt{3}$ D. $3\sqrt{4}$

If you missed it · 如果做错了 — **Perfect squares and square roots**

完全平方数与平方根

: Know the squares $1,4,9,\ldots,225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

记住 $1,4,9,\ldots,225$ 这些平方数。$\sqrt{ab}=\sqrt{a}\sqrt{b}$，所以 $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$。但 $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$：$\sqrt{9+16}=5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2=49$ has TWO answers $x=\pm 7$, but $\sqrt{49}=7$ only.
   $x^2=49$ 有两个解 $x=\pm 7$，但 $\sqrt{49}$ 只等于 7。

4. Simplify $\dfrac{6x+9}{3}$.

> 化简 $\dfrac{6x+9}{3}$。

A. $2x+9$ B. $2x+3$ C. $6x+3$ D. $3x+3$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
   > $\frac{2}{3}-\frac{1}{4}$：公分母 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$。
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
   > $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$。
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$。

Answers · 答案: 1: A · 2: C · 3: B · 4: B

🔍 Common-errors sheet

常见错误单

1. Adding radicals by adding the insides

> **根式相加时把根号里的数相加**

Simplify $\sqrt{8}+\sqrt{18}$.

> 化简 $\sqrt{8}+\sqrt{18}$。

1 $\sqrt{8}+\sqrt{18}=\sqrt{26}$

2 $26=2\cdot 13$ has no square factor, so the answer is $\sqrt{26}$.

> $26=2\cdot 13$ 没有平方因数，所以答案是 $\sqrt{26}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Squaring a binomial without the middle term

> **二项式平方漏了中间项**

Expand $(\sqrt{5}-2)^2$.

> 展开 $(\sqrt{5}-2)^2$。

1 $(\sqrt{5}-2)^2=(\sqrt{5})^2+(-2)^2$

2 $=5+4=9$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Multiplying only the bottom by the conjugate

> **只给分母乘共轭式**

Rationalize $\dfrac{2}{\sqrt{6}+2}$.

> 把 $\dfrac{2}{\sqrt{6}+2}$ 的分母有理化。

1 Multiply the bottom by $\sqrt{6}-2$: $(\sqrt{6}+2)(\sqrt{6}-2)=6-4=2$

> 分母乘 $\sqrt{6}-2$：$(\sqrt{6}+2)(\sqrt{6}-2)=6-4=2$

2 So the fraction is $\dfrac{2}{2}=1$.

> 所以分式等于 $\dfrac{2}{2}=1$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Forgetting to square the number in front of the root

> **忘了把根号前的系数平方**

Rationalize $\dfrac{14}{3\sqrt{2}-2}$.

> 把 $\dfrac{14}{3\sqrt{2}-2}$ 的分母有理化。

1 Multiply top and bottom by $3\sqrt{2}+2$.

> 分子分母同乘 $3\sqrt{2}+2$。

2 Bottom: $(3\sqrt{2})^2 - 2^2 = 6 - 4 = 2$

> 分母：$(3\sqrt{2})^2 - 2^2 = 6 - 4 = 2$

3 $\dfrac{14(3\sqrt{2}+2)}{2} = 21\sqrt{2} + 14$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Dividing only one term of the top

> **约分时只除了分子的一项**

Simplify $\dfrac{12+4\sqrt{5}}{4}$.

> 化简 $\dfrac{12+4\sqrt{5}}{4}$。

1 $12 \div 4 = 3$

2 So the answer is $3+4\sqrt{5}$.

> 所以答案是 $3+4\sqrt{5}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 1 —** Adding the insides feels like the same move as multiplying them. Only like radicals add, and only their outside numbers: like $2a + 3a = 5a$.

   > 感觉相加和相乘一样，把里面的数直接合起来就行。其实只有同类根式能加，而且加的是外面的系数：就像 $2a + 3a = 5a$。

   Simplify first: $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$. Decimal check: $2.83 + 4.24 = 7.07 = 5\sqrt{2}$, not $\sqrt{26} \approx 5.10$.

   > 先化简：$2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$。小数检验：$2.83 + 4.24 = 7.07 = 5\sqrt{2}$，而 $\sqrt{26} \approx 5.10$。

2. **line · 第 1 —** Squaring each piece feels like the power rule $(ab)^2 = a^2b^2$, but that works for products, not sums. $(A - B)^2$ means $(A - B)(A - B)$: four products.

   > 把每一部分分别平方，像是用了 $(ab)^2 = a^2b^2$；可那只对乘积成立，对和差不成立。$(A - B)^2$ 就是 $(A - B)(A - B)$，要乘出四项。

   $5 - 2 \cdot 2\sqrt{5} + 4 = 9 - 4\sqrt{5}$.

   > $5 - 2 \cdot 2\sqrt{5} + 4 = 9 - 4\sqrt{5}$。

3. **line · 第 1 —** The focus is on getting rid of the root below, so the top is forgotten. Multiplying only the bottom changes the value; top and bottom together is multiplying by 1.

   > 一心想消掉分母的根号，就忘了分子。只乘分母会改变值；分子分母一起乘才等于乘 1。

   $\dfrac{2(\sqrt{6} - 2)}{2} = \sqrt{6} - 2 \approx 0.449$. Check: $2 \div 4.449 \approx 0.449$, not 1.

   > $\dfrac{2(\sqrt{6} - 2)}{2} = \sqrt{6} - 2 \approx 0.449$。检验：$2 \div 4.449 \approx 0.449$，不是 1。

4. **line · 第 2 —** $(3\sqrt{2})^2$ was read as "3 times 2". The 3 is squared too: $(3\sqrt{2})^2 = 9 \cdot 2 = 18$.

   > 把 $(3\sqrt{2})^2$ 看成「3 乘 2」。系数 3 也要平方：$(3\sqrt{2})^2 = 9 \cdot 2 = 18$。

   Bottom: $18 - 4 = 14$, so $\dfrac{14(3\sqrt{2} + 2)}{14} = 3\sqrt{2} + 2$.

   > 分母：$18 - 4 = 14$，所以 $\dfrac{14(3\sqrt{2} + 2)}{14} = 3\sqrt{2} + 2$。

5. **line · 第 2 —** The 4 below divides the WHOLE top. Cancelling with the first number only quietly turns the question into $\dfrac{12}{4} + 4\sqrt{5}$.

   > 下面的 4 要除「整个」分子。只和第一个数约分，等于偷偷把题目改成了 $\dfrac{12}{4} + 4\sqrt{5}$。

   Divide both terms: $\dfrac{12}{4} + \dfrac{4\sqrt{5}}{4} = 3 + \sqrt{5}$.

   > 两项都除：$\dfrac{12}{4} + \dfrac{4\sqrt{5}}{4} = 3 + \sqrt{5}$。