Radical equations & extraneous roots

Thinking map — keep it next to you while you work

🧭 Five questions before you write

1. 🔎 **Given** What is under the root, and what is on the other side? Write the restriction: inside ≥ 0, and if the other side has x, it must be ≥ 0 too.
2. 🎯 **Asked** Every x that makes the ORIGINAL equation true. There may be two, one or none.
3. ✏️ **Picture it** Two graphs, $y =$ left side and $y =$ right side; answers are where they meet. A root curve can never meet a negative number.
4. 🧰 **Tool** Isolate the radical, square BOTH WHOLE sides, then solve the linear or quadratic equation you get.
5. ✅ **Check** Put every answer back into the ORIGINAL equation. Mark ✓ or ×; reject the × ones (extraneous).

⭐ Must know

- Restrictions → ISOLATE the radical → raise to the index → solve → CHECK in the original.
- Squaring can create extraneous roots: reject any root that fails the original.
- $\sqrt{\dots} \geq 0$: so $\sqrt{x} = -4$ has no solution, and $\sqrt{\dots} = x - 3$ needs $x \geq 3$.

$$\sqrt{A} = B \;\Rightarrow\; A = B^2, \quad \text{valid only if } B \geq 0$$

🔍 Watch out for

1. Squaring before isolating
2. Keeping the extraneous root
3. Squaring a binomial without the middle term
4. Forgetting to square the coefficient
5. Solving "root = negative number"

🌱 Small wins — tick as you go

- ○ I can expand $(x - a)^2$ and solve a quadratic by factoring. (5′)
- ○ I can ask the five starting questions for a radical equation. (5′)
- ○ I can follow, then finish, an isolate–square–check solution. (8′)
- ○ I can solve one on my own with at most two hints. (8′)
- ○ I can spot the five classic mistakes in someone else's work. (8′)
- ○ I can do the Level 1–2 practice questions. (10′)

Prerequisite repair sheet

Do the question first. Missed it? Read the rule and the worked lines under it, then try again.

1. Expand $(x-3)^2$.
A. x^2-9 B. x^2+9 C. x^2-6x+9 D. x^2-3x+9

If you missed it — **Multiplying binomials, $(a+b)^2$**: Every term times every term (FOIL): $(x+3)(x-5)=x^2-5x+3x-15=x^2-2x-15$. A square has a middle term: $(x+3)^2=x^2+6x+9$, never x^2+9.

1. $(x-4)^2=(x-4)(x-4)=x^2-4x-4x+16=x^2-8x+16$.
2. Shortcut: $(a\pm b)^2=a^2\pm 2ab+b^2$ — middle term $=2\times$ first $\times$ last.

2. Solve $x^2-7x+10=0$.
A. $x=-2$ or $x=-5$ B. $x=2$ or $x=5$ C. $x=1$ or $x=10$ D. $x=7$ or $x=10$

If you missed it — **Factoring x^2+bx+c (product and sum)**: Take out a common factor first. Then find two numbers that MULTIPLY to c and ADD to b: x^2-x-12 → -4 and 3 → $(x-4)(x+3)$. Check by expanding.

1. $x^2+7x+10$: $2\times 5=10$, $2+5=7$ → $(x+2)(x+5)$.
2. If $c<0$ the two numbers have different signs; if $c>0$ they share the sign of b.

3. What is $\sqrt{9+16}$?
A. 7 B. 5 C. 25 D. $\sqrt{7}$

If you missed it — **Perfect squares and square roots**: Know the squares $1,4,9,\ldots,225$. $\sqrt{ab}=\sqrt{a}\sqrt{b}$, so $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$. But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$: $\sqrt{9+16}=5$, not 7. $\sqrt{x}$ means the non-negative root.

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
2. $x^2=49$ has TWO answers $x=\pm 7$, but $\sqrt{49}=7$ only.

4. Evaluate $\sqrt{x+5}-x$ when $x=-1$.
A. 1 B. 3 C. -3 D. 5

If you missed it — **Negative numbers: add, subtract, multiply**: Subtracting a negative is adding: $5-(-3)=5+3=8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2=16$, but $-4^2=-16$ (the square happens first).

1. $-7-(-2)$: change "minus negative" to "plus": $-7+2=-5$.
2. $(-3)(-5)=15$ (same signs), $(-3)(5)=-15$ (different signs).
3. Plugging in $x=-3$ to x^2: write brackets first, $(-3)^2=9$.

Answers: 1: C · 2: B · 3: B · 4: B

🔍 Common-errors sheet

1. Squaring before isolating

Solve $\sqrt{x+5}+1=x$.

1 Square each term: $(x+5)+1=x^2$

2 $x^2-x-6=0$

3 $x=3$ or $x=-2$

✍ Which line goes wrong first, and why?

2. Keeping the extraneous root

Solve $\sqrt{2x+6}=x+1$.

1 Square: $2x+6=x^2+2x+1$

2 $x^2=5$

3 $x=\sqrt{5}$ or $x=-\sqrt{5}$, both answers.

✍ Which line goes wrong first, and why?

3. Squaring a binomial without the middle term

Solve $\sqrt{x+4}=x-2$.

1 Square: $x+4=x^2+4$

2 $x^2-x=0$

3 $x=0$ or $x=1$

✍ Which line goes wrong first, and why?

4. Forgetting to square the coefficient

Solve $3\sqrt{x-1} = x+1$.

1 Square: $3(x-1) = (x+1)^2$

2 $x^2 - x + 4 = 0$

3 No real solution.

✍ Which line goes wrong first, and why?

5. Solving "root = negative number"

Solve $\sqrt{x-3} + 5 = 2$.

1 Isolate: $\sqrt{x-3} = -3$

2 Square: $x - 3 = 9$

3 $x = 12$

✍ Which line goes wrong first, and why?

Key (fold this part back)

1. **line 1 —** You cannot square term by term. $(\sqrt{x+5}+1)^2$ has a middle term $2\sqrt{x+5}$. Isolate first: $\sqrt{x+5} = x-1$, then $x+5 = x^2 - 2x + 1$, $x^2 - 3x - 4 = 0$, $x = 4$ or -1; only $x = 4$ checks.
2. **line 3 —** Squaring can create a fake root. At $x = -\sqrt{5}$ the right side $x+1$ is negative, but a square root never is. Check both in the original: only $x = \sqrt{5}$ works.
3. **line 1 —** $(x-2)^2$ is $x^2 - 4x + 4$, not $x^2 + 4$. $x + 4 = x^2 - 4x + 4$, $x^2 - 5x = 0$, $x = 0$ or 5; only $x = 5$ checks.
4. **line 1 —** $(3\sqrt{x-1})^2 = 9(x-1)$: the 3 is squared too. $9x - 9 = x^2 + 2x + 1$, $x^2 - 7x + 10 = 0$, $x = 2$ or 5; both check ($3 = 3$, $6 = 6$).
5. **line 2 —** A square root is never negative, so $\sqrt{x-3} = -3$ is impossible. Squaring hides that. Stop at line 1: no solution. (Check: $\sqrt{9} + 5 = 8 \neq 2$.)