Rational exponents

分数指数

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Circle three things: the base, the denominator of the exponent (the root) and the numerator (the power). Note any minus sign in the exponent.
   > 圈出三样东西：底数、指数的分母（开几次方）、指数的分子（几次幂）。指数上有负号也标出来。
2. 🎯 **Asked · 求什么** Is it "evaluate" (a number), "simplify" (one power of each variable, positive exponents) or "write as a radical"? The answer form depends on it.
   > 题目要的是「求值」（一个数）、「化简」（每个字母只剩一个幂，指数为正），还是「写成根式」？答案形式由它决定。
3. ✏️ **Picture it · 画出来** Write the base as a power of a small number: $64 = 2^6 = 4^3 = 8^2$. Seeing that list tells you which root comes out clean.
   > 把底数写成小数字的幂：$64 = 2^6 = 4^3 = 8^2$。看着这串式子，就知道开几次方能开得干净。
4. 🧰 **Tool · 用什么工具** Flip first if the exponent is negative, then take the root, then the power. For variables, use the exponent laws and add/subtract the fractions.
   > 指数是负的就先翻转（取倒数），再开方，再乘方。字母部分用指数律，对分数指数做加减。
5. ✅ **Check · 检验** A negative exponent never makes the answer negative. Check backwards: raise your answer to the reciprocal power and see if you get the start again.
   > 负指数不会让结果变负。倒推检验：把答案取倒数次幂，看能不能回到原来的数。

⭐ Must know

必须记住

- $a^{m/n} = (\sqrt[n]{a})^m$: bottom of the exponent = root, top = power.
  > $a^{m/n} = (\sqrt[n]{a})^m$：指数的分母是开方次数，分子是乘方次数。
- By hand, take the root first: $27^{2/3} = 3^2 = 9$.
  > 手算先开方，数字小：$27^{2/3} = 3^2 = 9$。
- Negative exponent = reciprocal, never a negative answer: $16^{-1/2} = \frac{1}{4}$.
  > 负指数＝倒数，结果不会变负：$16^{-1/2} = \frac{1}{4}$。

$$a^{\frac{1}{n}} = \sqrt[n]{a}, \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$$

🔍 Watch out for

当心

1. Reading the root and the power the wrong way round
   > 把开方和乘方读反了
2. A negative exponent gives a negative answer
   > 以为负指数让结果变负
3. Multiplying the base by the exponent
   > 把底数和指数相乘
4. Multiplying exponents when you should add them
   > 该把指数相加，却相乘了
5. Raising only the variable, not the number in front
   > 只给字母乘方，漏了前面的数字

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can use $x^a \cdot x^b = x^{a+b}$, find small cube roots, and add fractions like $\frac{2}{3} + \frac{1}{4}$.
  > 我会用 $x^a \cdot x^b = x^{a+b}$，会求小数字的立方根，会算 $\frac{2}{3} + \frac{1}{4}$ 这样的分数加法。

  (5′)
- ○ I can read $a^{m/n}$ out loud as "the nth root, then the mth power" and say what a minus sign in the exponent does.
  > 我能把 $a^{m/n}$ 读成「先开 n 次方，再 m 次方」，也能说出指数上的负号起什么作用。

  (5′)
- ○ I can follow, then finish, the flip–root–power method for $a^{-m/n}$.
  > 我能看懂并补全 $a^{-m/n}$ 的「翻转—开方—乘方」三步法。

  (8′)
- ○ I can evaluate a negative rational power of a fraction on my own with at most two hints.
  > 我能在最多两个提示下，独立算出分数的负分数指数幂。

  (6′)
- ○ I can spot the five classic rational-exponent mistakes in someone else's work.
  > 我能在别人的解答里找出分数指数的五个经典错误。

  (8′)
- ○ I can do the Level 1–2 practice questions without a calculator.
  > 我能不用计算器做出 1–2 级练习题。

  (8′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Simplify $x^3 \cdot x^4$.

> 化简 $x^3 \cdot x^4$。

A. x^{12} B. x^7 C. $2x^7$ D. x^{34}

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

: Same base: multiply → ADD exponents ($x^3x^4 = x^7$); divide → SUBTRACT ($x^7 \div x^2 = x^5$); power of a power → MULTIPLY ($(x^3)^2 = x^6$). A power of a product hits every factor: $(2x)^3 = 8x^3$.

> 同底数：相乘→指数相加（$x^3x^4 = x^7$）；相除→指数相减（$x^7 \div x^2 = x^5$）；幂的乘方→指数相乘（$(x^3)^2 = x^6$）。积的乘方每个因数都要乘方：$(2x)^3 = 8x^3$。

1. $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$.
   > $(3x^2)^2 \cdot x = 9x^4 \cdot x = 9x^5$。
2. $\frac{6x^5}{2x^2} = 3x^3$ (divide the numbers, subtract the exponents).
   > $\frac{6x^5}{2x^2} = 3x^3$（系数相除，指数相减）。

2. What is $\sqrt[3]{64}$?

> $\sqrt[3]{64}$ 等于多少?

A. 4 B. 8 C. $\frac{64}{3}$ D. 16

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

3. Work out $\frac{2}{3} + \frac{1}{4}$.

> 计算 $\frac{2}{3} + \frac{1}{4}$。

A. $\frac{3}{7}$ B. $\frac{2}{12}$ C. $\frac{5}{12}$ D. $\frac{11}{12}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3} - \frac{1}{4}$: common denominator 12 → $\frac{8}{12} - \frac{3}{12} = \frac{5}{12}$.
   > $\frac{2}{3} - \frac{1}{4}$：公分母 12 → $\frac{8}{12} - \frac{3}{12} = \frac{5}{12}$。
2. $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$.
   > $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{24}{36} = \frac{2}{3}$。
3. Never add denominators: $\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$。

4. What is 2^{-3}?

> 2^{-3} 等于多少?

A. -8 B. -6 C. $\frac{1}{8}$ D. $\frac{1}{6}$

If you missed it · 如果做错了 — **Exponent rules (product, quotient, power)**

> **指数运算法则（同底乘、除、乘方）**

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Answers · 答案: 1: B · 2: A · 3: D · 4: C

🔍 Common-errors sheet

常见错误单

1. Reading the root and the power the wrong way round

把开方和乘方读反了

Evaluate $64^{2/3}$.

求 $64^{2/3}$。

1 Square root of 64: $\sqrt{64} = 8$

64 开平方：$\sqrt{64} = 8$

2 Cube it: $8^3 = 512$

再立方：$8^3 = 512$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. A negative exponent gives a negative answer

以为负指数让结果变负

Evaluate $25^{-1/2}$.

求 $25^{-1/2}$。

1 $\sqrt{25} = 5$

2 The exponent is negative, so the answer is -5.

指数是负的，所以结果是 -5。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Multiplying the base by the exponent

把底数和指数相乘

Evaluate $27^{1/3}$.

求 $27^{1/3}$。

1 $27^{1/3} = 27 \times \frac{1}{3}$

2 $= 9$

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Multiplying exponents when you should add them

> **该把指数相加，却相乘了**

Simplify $x^{1/2} \cdot x^{1/4}$, $x > 0$.

> 化简 $x^{1/2} \cdot x^{1/4}$ $(x > 0)$。

1 Same base, so $x^{\frac{1}{2} \cdot \frac{1}{4}}$

> 同底数，所以 $x^{\frac{1}{2} \cdot \frac{1}{4}}$

2 $= x^{1/8}$

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Raising only the variable, not the number in front

> **只给字母乘方，漏了前面的数字**

Simplify $(16x^8)^{3/4}$, $x > 0$.

> 化简 $(16x^8)^{3/4}$ $(x > 0)$。

1 $x^{8 \cdot \frac{3}{4}} = x^6$

2 So $(16x^8)^{3/4} = 16x^6$

> 所以 $(16x^8)^{3/4} = 16x^6$

✍️ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 1 —** The 2 on top looked like "square root". But the DENOMINATOR is the root: $\frac{2}{3}$ means cube root, then square.

   > 分子上的 2 看着像「开平方」。其实分母才是开方次数：$\frac{2}{3}$ 表示先开立方，再平方。

   $\sqrt[3]{64} = 4$, then $4^2 = 16$.

   > $\sqrt[3]{64} = 4$，再 $4^2 = 16$。

2. **line · 第 2 —** The minus sign in an exponent means "take the reciprocal", not "make it negative". A positive base always gives a positive power.

   > 指数上的负号表示「取倒数」，不是「变成负数」。底数为正，幂永远为正。

   $25^{-1/2} = \dfrac{1}{25^{1/2}} = \dfrac{1}{5}$.

   > $25^{-1/2} = \dfrac{1}{25^{1/2}} = \dfrac{1}{5}$。

3. **line · 第 1 —** An exponent is not a multiplier. $\frac{1}{3}$ as an exponent asks "what number cubed gives 27?"

   > 指数不是乘数。指数 $\frac{1}{3}$ 问的是「哪个数立方等于 27」。

   $27^{1/3} = \sqrt[3]{27} = 3$, since $3^3 = 27$.

   > $27^{1/3} = \sqrt[3]{27} = 3$，因为 $3^3 = 27$。

4. **line · 第 1 —** Fractions made the familiar rule feel new. Multiplying powers of the same base still means ADD the exponents; you multiply only for a power of a power.

   > 一碰到分数，熟悉的法则就忘了。同底数幂相乘仍然是指数「相加」；只有幂的乘方才是指数相乘。

   $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$, so $x^{3/4}$.

   > $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$，所以是 $x^{3/4}$。

5. **line · 第 2 —** The 16 looked like a plain coefficient that just comes along. But the bracket means every factor inside is raised to $\frac{3}{4}$.

   > 16 看起来像一个跟着走的系数。可是括号表示里面每个因数都要取 $\frac{3}{4}$ 次方。

   $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$, so $8x^6$.

   > $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$，所以是 $8x^6$。