The real number system

实数系统

Thinking map — keep it next to you while you work · 思路图——做题时放在手边

🧭 Five questions before you write

动笔前的五个问题

1. 🔎 **Given · 已知** Write each number exactly as given. Note what it has: a root (which root, and of what), a fraction, a repeating bar, π.
   > 把每个数照原样写下来，看它带着什么：根号（几次方根、根号下是什么）、分数、循环节，还是 π。
2. 🎯 **Asked · 求什么** Which sets does it belong to? List ALL of them from $\mathbb{N}$, $\mathbb{W}$, $\mathbb{I}$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{R}$, or say "not real".
   > 它属于哪些数集？从 $\mathbb{N}$、$\mathbb{W}$、$\mathbb{I}$、$\mathbb{Q}$、$\overline{\mathbb{Q}}$、$\mathbb{R}$ 中写出「全部」，或者写「不是实数」。
3. ✏️ **Picture it · 画出来** Nested boxes: $\mathbb{N}$ inside $\mathbb{W}$ inside $\mathbb{I}$ inside $\mathbb{Q}$, with $\overline{\mathbb{Q}}$ beside $\mathbb{Q}$, all inside $\mathbb{R}$. Find the smallest box the number fits; it is also in every box around that one.
   > 一层套一层的盒子：$\mathbb{N}$ 在 $\mathbb{W}$ 里，$\mathbb{W}$ 在 $\mathbb{I}$ 里，$\mathbb{I}$ 在 $\mathbb{Q}$ 里；$\overline{\mathbb{Q}}$ 和 $\mathbb{Q}$ 并排，都在 $\mathbb{R}$ 里。找出装得下这个数的最小盒子，外面套着它的每个盒子也都装着它。
4. 🧰 **Tool · 用什么工具** Simplify first: work out roots of perfect squares and cubes, reduce fractions, turn repeating decimals into fractions. An even root of a negative is not real, so stop there.
   > 先化简：开得尽的平方根、立方根直接算出来，分数约分，循环小数化成分数。负数开偶次方不是实数，到此为止。
5. ✅ **Check · 检验** Decimal test: it ends or repeats, so rational; it never ends and never repeats, so irrational. Then make sure you listed every bigger set too.
   > 小数检验：有限或循环，就是有理数；无限不循环，就是无理数。最后确认更大的数集也都写上了。

⭐ Must know

必须记住

- $\mathbb{N} \subset \mathbb{W} \subset \mathbb{I} \subset \mathbb{Q} \subset \mathbb{R}$; 0 is whole but not natural.
  > $\mathbb{N} \subset \mathbb{W} \subset \mathbb{I} \subset \mathbb{Q} \subset \mathbb{R}$；$0$ 是非负整数（$\mathbb{W}$），但不是自然数。
- Rational = $\frac{a}{b}$ with integers a, b: decimal ends or repeats. Irrational: never ends, never repeats.
  > 有理数＝两个整数之比 $\frac{a}{b}$，小数会终止或循环；无理数的小数不终止也不循环。
- Simplify before classifying: $\sqrt{49} = 7$, $\sqrt[3]{-8} = -2$; $\sqrt{-4}$ is not real.
  > 先化简再分类：$\sqrt{49} = 7$，$\sqrt[3]{-8} = -2$；$\sqrt{-4}$ 不是实数。

$$\mathbb{N} \subset \mathbb{W} \subset \mathbb{I} \subset \mathbb{Q} \subset \mathbb{R}, \qquad \mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$$

$$0.\overline{ab} = \frac{ab}{99}$$

🔍 Watch out for

当心

1. Listing only the smallest set
   > 只写了最小的那个数集
2. Calling a number irrational because of the root sign
   > 看到根号就说是无理数
3. Thinking any fraction shape is rational
   > 以为长得像分数就是有理数
4. Turning an even root of a negative into a negative number
   > 把负数的偶次方根当成负数
5. Deciding from the calculator screen
   > 看计算器屏幕下结论

🌱 Small wins — tick as you go

一小步一小步——做完打勾

- ○ I can spot perfect squares and cubes and reduce a fraction.
  > 我能认出完全平方数和完全立方数，也会约分。

  (5′)
- ○ I can ask the five starting questions before I classify a number.
  > 给一个数分类之前，我能先问出五个入手问题。

  (5′)
- ○ I can follow, then finish, a simplify–test–climb classification.
  > 我能看懂并补全「先化简—再检验—逐级往上」的分类过程。

  (8′)
- ○ I can classify three new numbers on my own with at most two hints.
  > 我能在最多两个提示下，独立给三个新的数分类。

  (6′)
- ○ I can spot the five classic classification mistakes in someone else's work.
  > 我能在别人的解答里找出五个经典的分类错误。

  (8′)
- ○ I can do the Level 1–2 practice questions.
  > 我能做出 1–2 级练习题。

  (8′)

Prerequisite repair sheet

先修补救单

Do the question first. Missed it? Read the rule and the worked lines under it, then try again. · 先做题。做错了？读下面的规则和例子，再做一遍。

1. Which number is a perfect square?

> 下面哪个数是完全平方数?

A. 50 B. 64 C. 80 D. 90

If you missed it · 如果做错了 — **Perfect squares and square roots**

> **完全平方数与平方根**

: Know the squares $1, 4, 9, \ldots, 225$. $\sqrt{ab} = \sqrt{a}\sqrt{b}$, so $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$. But $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$, not 7. $\sqrt{x}$ means the non-negative root.

> 记住 $1, 4, 9, \ldots, 225$ 这些平方数。$\sqrt{ab} = \sqrt{a}\sqrt{b}$，所以 $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$。但 $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$：$\sqrt{9+16} = 5$，不是 7。$\sqrt{x}$ 表示非负的那个根。

1. $\sqrt{72}$: biggest square factor is 36 → $6\sqrt{2}$.
   > $\sqrt{72}$：最大的平方因数是 36 → $6\sqrt{2}$。
2. $x^2 = 49$ has TWO answers $x = \pm 7$, but $\sqrt{49} = 7$ only.
   > $x^2 = 49$ 有两个解 $x = \pm 7$，但 $\sqrt{49}$ 只等于 7。

2. Work out $(-3)^3$.

> 计算 $(-3)^3$。

A. -27 B. 27 C. -9 D. 9

If you missed it · 如果做错了 — **Negative numbers: add, subtract, multiply**

> **负数的加减乘**

: Subtracting a negative is adding: $5 - (-3) = 5 + 3 = 8$. Same signs multiply to $+$, different signs to $-$. Squaring a negative is positive: $(-4)^2 = 16$, but $-4^2 = -16$ (the square happens first).

> 减去负数等于加上它：$5 - (-3) = 5 + 3 = 8$。同号相乘得正，异号相乘得负。负数的平方是正的：$(-4)^2 = 16$，但 $-4^2 = -16$（先算平方）。

1. $-7 - (-2)$: change "minus negative" to "plus": $-7 + 2 = -5$.
   > $-7 - (-2)$：把「减负」改成「加」：$-7 + 2 = -5$。
2. $(-3)(-5) = 15$ (same signs), $(-3)(5) = -15$ (different signs).
   > $(-3)(-5) = 15$（同号），$(-3)(5) = -15$（异号）。
3. Plugging in $x = -3$ to x^2: write brackets first, $(-3)^2 = 9$.
   > 把 $x = -3$ 代入 x^2：先加括号，$(-3)^2 = 9$。

3. Write $\frac{36}{99}$ in lowest terms.

> 把 $\frac{36}{99}$ 化成最简分数。

A. $\frac{2}{9}$ B. $\frac{4}{11}$ C. $\frac{9}{11}$ D. $\frac{6}{33}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

> 加减必须先通分：$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$。乘法分子乘分子、分母乘分母。除以一个分数＝乘它的倒数（把第二个分数倒过来）。

1. $\frac{2}{3}-\frac{1}{4}$: common denominator 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.
   > $\frac{2}{3}-\frac{1}{4}$：公分母 12 → $\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$。
2. $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$.
   > $\frac{3}{4}\div\frac{9}{8}=\frac{3}{4}\cdot\frac{8}{9}=\frac{24}{36}=\frac{2}{3}$。
3. Never add denominators: $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$.
   > 分母不能相加：$\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$。

4. Write 0.75 as a fraction in lowest terms.

> 把 0.75 写成最简分数。

A. $\frac{3}{5}$ B. $\frac{7}{5}$ C. $\frac{3}{4}$ D. $\frac{75}{10}$

If you missed it · 如果做错了 — **Fractions: add, subtract, multiply, divide**

> **分数的四则运算**

: Add or subtract only with a common denominator: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$. Multiply straight across. Divide = multiply by the reciprocal (flip the second fraction).

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   > 分母不能相加：$\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$。

Answers · 答案: 1: B · 2: A · 3: B · 4: C

🔍 Common-errors sheet

常见错误单

1. Listing only the smallest set

只写了最小的那个数集

Name every set that $\sqrt{81}$ belongs to.

写出 $\sqrt{81}$ 所属的全部数集。

1 $\sqrt{81} = 9$.

$\sqrt{81} = 9$。

2 9 is a counting number, so the answer is $\mathbb{N}$.

9 是用来数数的数，所以答案是 $\mathbb{N}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

2. Calling a number irrational because of the root sign

看到根号就说是无理数

Is $\sqrt{0.09}$ rational or irrational?

$\sqrt{0.09}$ 是有理数还是无理数?

1 It has a root sign, and 0.09 is not a perfect square like 4 or 9.

它带根号，而且 0.09 不是像 4、9 那样的完全平方数。

2 So $\sqrt{0.09}$ is irrational.

所以 $\sqrt{0.09}$ 是无理数。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

3. Thinking any fraction shape is rational

以为长得像分数就是有理数

Is $\dfrac{\sqrt{3}}{2}$ rational or irrational?

$\dfrac{\sqrt{3}}{2}$ 是有理数还是无理数?

1 It is written as a top over a bottom.

它写成了分子在上、分母在下的样子。

2 A fraction is rational, so $\dfrac{\sqrt{3}}{2}$ is rational.

分数是有理数，所以 $\dfrac{\sqrt{3}}{2}$ 是有理数。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

4. Turning an even root of a negative into a negative number

> **把负数的偶次方根当成负数**

Classify $\sqrt{-16}$.

> 给 $\sqrt{-16}$ 分类。

1 $\sqrt{-16} = -4$.

> $\sqrt{-16} = -4$。

2 -4 is an integer, so $\mathbb{I}, \mathbb{Q}, \mathbb{R}$.

> -4 是整数，所以属于 $\mathbb{I}, \mathbb{Q}, \mathbb{R}$。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

5. Deciding from the calculator screen

> **看计算器屏幕下结论**

Is $\frac{7}{13}$ rational or irrational?

> $\frac{7}{13}$ 是有理数还是无理数?

1 The calculator shows 0.5384615385.

> 计算器显示 0.5384615385。

2 No repeat shows on the screen, so it is irrational.

> 屏幕上看不出循环，所以是无理数。

✍ Which line goes wrong first, and why? · 哪一行最先出错？为什么？

Key (fold this part back) · 答案（可以折起来）

1. **line · 第 2 —** Finding the first set that fits feels like the end of the job. But the sets are nested, so 9 also sits in every bigger set.
   > 找到第一个装得下它的数集，就以为做完了。可是数集一层套一层，9 也在每个更大的数集里。

   Climb the ladder: $9 \in \mathbb{N}, \mathbb{W}, \mathbb{I}, \mathbb{Q}, \mathbb{R}$.
   > 沿梯子往上写：$9 \in \mathbb{N}, \mathbb{W}, \mathbb{I}, \mathbb{Q}, \mathbb{R}$。
2. **line · 第 1 —** Only whole numbers were tried as perfect squares. Decimals can be perfect squares too: $0.3^2 = 0.09$.
   > 只拿整数去想完全平方数。小数也可以是完全平方：$0.3^2 = 0.09$。

   $\sqrt{0.09} = 0.3 = \dfrac{3}{10}$, a ratio of integers, so it is rational.
   > $\sqrt{0.09} = 0.3 = \dfrac{3}{10}$，是两个整数之比，所以是有理数。
3. **line · 第 2 —** Rational means a ratio of two INTEGERS. The shape matches, but the top, $\sqrt{3}$, is not an integer.
   > 有理数的意思是两个「整数」之比。形状对上了，但分子 $\sqrt{3}$ 不是整数。

   $\sqrt{3}$ is irrational, and half of an irrational number is still irrational.
   > $\sqrt{3}$ 是无理数，无理数的一半仍是无理数。
4. **line · 第 1 —** It copies the cube-root habit ($\sqrt[3]{-8} = -2$). But $(-4)^2 = 16$, not -16: no real number squares to a negative.
   > 照搬了立方根的习惯（$\sqrt[3]{-8} = -2$）。可是 $(-4)^2 = 16$，不是 -16：没有实数的平方是负数。

   Stop at line 1: $\sqrt{-16}$ is not a real number, so it is in none of the sets.
   > 在第 1 行就停：$\sqrt{-16}$ 不是实数，不属于任何一个数集。
5. **line · 第 2 —** The screen only has room for about ten digits. The block 538461 has six digits, so its repeat is cut off by rounding.
   > 屏幕只放得下十位左右。循环节 538461 有六位，它的第二轮被四舍五入截掉了。

   Decide from the exact form: $\dfrac{7}{13}$ is a ratio of integers, so it is rational ($0.\overline{538461}$).
   > 按精确形式判断：$\dfrac{7}{13}$ 是两个整数之比，所以是有理数（$0.\overline{538461}$）。